

BOUNDARY LAYERS AND INCOMPRESSIBLE NAVIER-STOKES-FOURIER LIMIT OF THE BOLTZMANN EQUATION IN BOUNDED DOMAIN (I)

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ABSTRACT. We establish the incompressible Navier-Stokes-Fourier limit for solutions to the Boltzmann equation with a general cut-off collision kernel in a bounded domain. Appropriately scaled families of DiPerna-Lions-(Mischler) renormalized solutions with Maxwell reflection boundary conditions are shown to have fluctuations that converge as the Knudsen number goes to zero. Every limit point is a weak solution to the Navier-Stokes-Fourier system with different types of boundary conditions depending on the ratio between the accommodation coefficient and the Knudsen number. The main new result of the paper is that this convergence is strong in the case of Dirichlet boundary condition. Indeed, we prove that the acoustic waves are damped immediately, namely they are damped in a boundary layer in time. This damping is due to the presence of viscous and kinetic boundary layers in space. As a consequence, we also justify the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with Navier-Stokes scaling.

This extends the work of Golse and Saint-Raymond [19, 20] and Levermore and Masmoudi [27] to the case of a bounded domain. The case of a bounded domain was considered by Masmoudi and Saint-Raymond [33] for linear Stokes-Fourier limit and Saint-Raymond [40] for Navier-Stokes limit for hard potential kernels. Both [33] and [40] didn't study the damping of the acoustic waves. This paper extends the result of [33] and [40] to the nonlinear case and includes soft potential kernels. More importantly, for the Dirichlet boundary condition, this work strengthens the convergence so as to make the boundary layer visible. This answers an open problem proposed by Ukai [45].

1. INTRODUCTION

The hydrodynamic limits from the Boltzmann equation got a lot of interest in the previous two decades. Hydrodynamic regimes are those where the Knudsen number ε is small. The Knudsen number is the ratio of the mean free path and the macroscopic length scales. The incompressible Navier-Stokes-Fourier (NSF) system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the number density F about an absolute Maxwellian M are scaled to be on the order ε , see [2].

The program that justifies the hydrodynamic limits from the Boltzmann equation in the framework of DiPerna-Lions [12] was initiated by Bardos-Golse-Levermore [2, 3] in late 80's. Since then, there has been lots of contributions to this program [4, 13, 19, 20, 23, 27, 30, 31, 33, 39]. In particular the work of Golse and Saint-Raymond [19] is the first complete rigorous justification of NSF limit from the Boltzmann equation in a class of bounded collision kernels, without making any nonlinear weak compactness hypothesis. They have recently extended their result to the case of hard potentials [20]. With some new nonlinear estimates, Levermore and Masmoudi [27] treated a broader class of collision kernels which includes all hard potential cases and, for the first time in this program, soft potential cases.

All of the above mentioned works were carried out in either the periodic spatial domain or the whole space, except for [33] and [40]. In [33], the linear Stokes-Fourier system was recovered with the same collision kernels assumption as in [13], while in [40], the Navier-Stokes limit was derived with the same kernels assumption as in [20], i.e. hard potential kernels. In [33] and [40], the fluctuations of renormalized solutions to the Boltzmann equation in a bounded domain (see

[37]) was proved to pass to the limit and recovered fluid boundary conditions, either Dirichlet, or Navier slip boundary condition, depending on the relative sizes of the accommodation coefficient and the Knudsen number.

The dependance of the boundary conditions of the limiting fluid equations on the relative importance of the accommodation coefficient and the Knudsen number was observed by Sone and his collaborators. Their results, mostly formal, are presented in Chapter 3 and 4 in [44] for several types of kinetic boundary conditions. The work [33] and [40] rigorously justified the incompressible Stokes and Navier-Stokes equations from Boltzmann equation imposed with Maxwell reflection boundary condition.

In his survey paper [45], Ukai proposed the following question: *“As far as the Boltzmann equation in a bounded domain is concerned, some progress has been made recently. In [37], the convergence of the Boltzmann equation to the (linear) Stokes-Fourier equation was proved together with the convergence of the boundary conditions. It is a big challenging problem to extend the result to the nonlinear case and to strength the convergence so as to make visible the boundary layer.”* (In the above citation of Ukai’s survey, the reference [37] is the Saint-Raymond and Masmoudi’s paper [33].)

In this paper and a forthcoming one, we study the incompressible NSF limit in a bounded domain from the Boltzmann equation with the Maxwell reflection boundary condition in which the accommodation might depend on the Knudsen number. We consider a bounded domain $\Omega \subset \mathbb{R}^D$, $D \geq 2$, with boundary $\partial\Omega \in C^2$. The NSF system governs the fluctuations of mass density, bulk velocity, and temperature $(\rho, \mathbf{u}, \theta)$ about their spatially homogeneous equilibrium values in a Boussinesq regime. Specifically, after a suitable choice of units, these dimensionless fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot \mathbf{u} = 0, \quad \rho + \theta = 0, \quad (1.1)$$

while their evolution is determined by the Navier-Stokes and heat equations

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p &= \nu \Delta_x \mathbf{u}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla_x \theta &= \frac{2}{D+2} \kappa \Delta_x \theta, \quad \theta|_{t=0} = \theta_0, \end{aligned} \quad (1.2)$$

where $\nu > 0$ is the kinematic viscosity and $\kappa > 0$ is the heat thermal conductivity.

Traditionally, two types of natural physical boundary conditions could be imposed for the incompressible NSF system (1.2). The first is the homogeneous Dirichlet boundary condition, namely,

$$\mathbf{u} = 0, \quad \theta = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \quad (1.3)$$

The other is the so-called Navier slip boundary condition, which was proposed by Navier [38]:

$$\begin{aligned} [2\nu d(\mathbf{u}) \cdot \mathbf{n} + \chi \mathbf{u}]^{\text{tan}} &= 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \kappa \partial_n \theta + \chi \frac{D+1}{D+2} \theta &= 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \quad (1.4)$$

where $d(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top)$ denotes the symmetric part of the stress tensor and ∂_n denotes the directional derivative along the outer normal vector $\mathbf{n}(x)$, $x \in \partial\Omega$. In the above Navier boundary condition, $\chi > 0$ is the reciprocal of the slip length which depends on the material of the container.

In the current work, for general cut-off collision kernels, namely in the framework of [27], we justify the NSF system. Regarding the *weak* convergence results, our proof is basically the same as in [33] and [40]: the boundary conditions of the limiting NSF system depend on the ratio of the accommodation coefficient and the Knudsen number, namely when $\frac{\alpha\varepsilon}{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, Dirichlet condition is derived, while when $\frac{\alpha\varepsilon}{\varepsilon} \rightarrow \sqrt{2\pi}\chi$, the Navier-slip boundary condition is derived. The main difference is that [40] used the same renormalizations of [20], applicable for hard potentials, while in the current work, we use the renormalization of [27], which works for more general cut-off kernels, including soft potentials.

The main novelty of the current work is the treatment of the Dirichlet boundary condition case. Indeed, we prove that when $\frac{\alpha\varepsilon}{\varepsilon} \rightarrow \infty$, the convergence is *strong*. Furthermore, as a consequence of this strong convergence, the first correction to the infinitesimal Maxwellian, which is a quadratic term obtained from the Chapman-Enskog expansion with the Navier-Stokes scaling, is rigorously justified. We point out that in all the previous works mentioned above, the convergence is in $w\text{-}L^1$, unless the initial data is well-prepared, i.e. is hydrodynamic and satisfies the Boussinesq and incompressibility relations. This weak convergence is caused by the persistence of fast acoustic waves. In the Navier-Stokes regime, the Reynold number Re is order $O(1)$, then the von Kármán relation $\varepsilon = \frac{Ma}{Re}$ implies that in the fluid limit $\varepsilon \rightarrow 0$, the Mach number Ma must go to zero. As is well known physically, one expects that as $Ma \rightarrow 0$, fast acoustic waves are generated and carry the energy of the potential part of the flow. For the periodic flows, or for some particular boundary conditions such as Navier condition (1.4), these waves subsist forever and their frequency grows with ε . Mathematically, this means that the convergence is only *weak*. This phenomenon happens in many singular limits of fluid equations among which we only mention [28, 29].

One of the ingredients of the convergence proof is the treatment of the acoustic waves which are highly oscillating. A compensated compactness type argument was used by Lions and Masmoudi [30] to prove that these acoustic waves have no contribution on the equation satisfied by the weak limit. This argument was previously used in the compressible incompressible limit [29].

In [10], a striking phenomenon, namely the damping of acoustic waves caused by the Dirichlet boundary condition was found by Desjardins, Grenier, Lions, and Masmoudi in considering the incompressible limit of the isentropic compressible Navier-Stokes equations. In the case of a viscous flow in a bounded domain with Dirichlet boundary condition, and under a generic assumption on the domain (related to the so-called Schiffer's conjecture and the Pompeiu problem [9]), they showed that the acoustic waves are instantaneously (asymptotically) damped, due to the formation of a thin boundary layer in time. This layer is caused by a boundary layer in space and dissipates the energy carried by the acoustic waves. From a mathematical point of view, strong convergence was obtained.

Inspired by the idea of [10], the current paper considers the much more involved kinetic-fluid coupled case. We prove that if the accommodation coefficient is bigger than the Knudsen number, there is no need for the argument in [29] since we can prove that the acoustic waves are damped instantaneously. Our work is based on the construction of viscous and kinetic Knudsen boundary layers of size $\sqrt{\varepsilon}$ and ε . The main idea is to use a family of test functions which solve approximately a scaled stationary linearized Boltzmann equation and can capture the propagation of the fast acoustic waves. These test functions are constructed through considering a family of approximate eigenfunctions of a *dual* operator with a *dual* kinetic boundary condition with respect to the original Boltzmann equation. The approximate eigenvalue is the sum of several terms with different order of ε : the leading term is purely imaginary, which describes the acoustic mode, and the real part of the next order term is strictly *negative* which gives the strict dissipation when applying the test functions to the renormalized Boltzmann equation.

In contrast to [10], the approximate eigenfunctions include interior part and two boundary layers: fluid viscous layer and kinetic Knudsen layer, while in [10], only a fluid boundary layer was necessary. Another important difference is that a generic assumption on the domain had to be made in [10] (in particular there are modes which are not damped in the disc), while in the current work, this assumption is not needed. The reason is that we deal with the full acoustic system, namely including the temperature. The NSF system has also some dissipation in the temperature equation which is ignored in the isentropic model. (in particular this dissipation property holds in the case of the ball). This was also considered in [24] in which we reinforced the result of [10].

When the accommodation coefficient α_ε is asymptotically larger than the Knudsen number ε in the sense that $\frac{\alpha_\varepsilon}{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, the fluid limit is the NSF equations with Dirichlet boundary condition. For example, we can assume $\alpha_\varepsilon = \chi \varepsilon^\beta$ with $0 \leq \beta < 1$. We found that $\beta = \frac{1}{2}$ is a threshold in the sense that the kinetic-fluid coupled boundary layers behave differently for $0 \leq \beta < \frac{1}{2}$ and $\frac{1}{2} \leq \beta < 1$, but for both cases the kinetic-fluid layers have damping effect. The current paper focuses on the threshold case $\beta = \frac{1}{2}$ and we leave the other cases for a separate paper due to the more complex construction of the boundary layers.

One of the difficulties of the construction happens in the case the Laplace operator $-\Delta_x$ with Neumann boundary condition has multiple eigenvalues. As a consequence, the dimension of the null space of the operator $\mathcal{A} - i\lambda_0^k$ is greater than one, where \mathcal{A} denotes the acoustic operator, and $\frac{D}{D+2}[\lambda_0^k]^2$ are eigenvalues for $k \in \mathbb{N}$ (for details see Section 5.2). Thus, as each stage of the construction of boundary layers, the terms in the null space of $\mathcal{A} - i\lambda_0^k$ can not be determined uniquely. To completely determine all the terms in the ansatz of boundary layers, we have to add some orthogonality conditions. Surprisingly, all these orthogonality conditions are consistent, at least for the threshold case $\beta = \frac{1}{2}$ treated in the current paper. Similar idea has been used in [24] which can be applied to the compressible-incompressible limit of the full Navier-Stokes-Fourier system in a bounded domain.

A key role is played by the linearized kinetic boundary layer equation in the coupling of viscous and kinetic layers. More specifically, its solvability provides the boundary conditions of the fluid variables in the interior and viscous boundary layers which satisfy the acoustic systems with source terms and second order ordinary differential equations respectively. This linearized kinetic boundary layer equation has been studied extensively (see [1, 8, 16, 15, 46]). Applying the boundary layer equations to construct the two layer eigenfunctions is the main novelty of the current paper. To the best of our knowledge these two layer eigenfunctions are new even in the applied literature.

The paper is organized as follows: the next section contains preliminary material regarding the Boltzmann equation in a bounded domain. We state the main theorems in Section 3 which include the weak convergence for the Navier slip boundary and strong convergence for the Dirichlet boundary. In Section 4, we list some differential geometry properties of the boundary $\partial\Omega$ as a submanifold of \mathbb{R}^D . Section 5 provides an introduction to the acoustic modes while Section 6 is about the analysis of the kinetic boundary layer equation whose solvability provides the boundary conditions of the fluid variables. In Section 7, we present the constructions of the test functions used in the proof of the Main Theorem. The proof of the main proposition on the boundary layers is given in Sections 8 and 9. In Section 10, we establish the weak convergence result of the main theorem. Section 11 contains the proof of the strong convergence in the Dirichlet boundary case using the test functions constructed in Section 7.

2. BOLTZMANN EQUATION IN BOUNDED DOMAIN

Here we introduce the Boltzmann equation in a bounded domain, only so far as to set our notations, which are essentially those of [3] and [33]. More complete introduction to the Boltzmann equation can be found in [6, 7, 17, 44].

2.1. Maxwell Boundary Condition. We consider Ω , a smooth bounded domain of \mathbb{R}^D , and $\mathcal{O} = \Omega \times \mathbb{R}^D$, the space-velocity domain. Let $\mathbf{n}(x)$ be the outward unit normal vector at $x \in \partial\Omega$ and let $d\sigma_x$ be the Lebesgue measure on the boundary $\partial\Omega$. We define the outgoing and incoming sets Σ_+ and Σ_- by

$$\Sigma_\pm = \{(x, v) \in \Sigma : \pm \mathbf{n}(x) \cdot v > 0\} \quad \text{where} \quad \Sigma = \partial\Omega \times \mathbb{R}^D.$$

Denoted by γF the trace of F over Σ , the boundary condition takes the form of a balance between the values of the outgoing and incoming parts of γF , namely $\gamma_\pm F = \mathbf{1}_{\Sigma_\pm} \gamma F$. In order

to describe the interaction between particles and the wall, Maxwell [34] proposed in 1879 the following phenomenological law which splits into a local reflection and a diffuse reflection

$$\gamma_- F = (1 - \alpha) L \gamma_+ F + \alpha K \gamma_+ F \quad \text{on } \Sigma_-, \quad (2.1)$$

where $\alpha \in [0, 1]$ is a constant, called the “accommodation coefficient.” The local reflection operator L is given by

$$L\phi(x, v) = \phi(x, R_x v), \quad (2.2)$$

where $R_x v = v - 2[n(x) \cdot v]n(x)$ is the velocity before the collision with the wall. The diffuse reflection operator K is given by

$$K\phi(x, v) = \sqrt{2\pi} \tilde{\phi}(x) M(v),$$

where $\tilde{\phi}$ is the outgoing mass flux

$$\tilde{\phi}(x) = \int_{v \cdot n(x) > 0} \phi(x, v) v \cdot n(x) dv,$$

and M is the absolute Maxwellian $M(v) = \frac{1}{(2\pi)^{D/2}} \exp(-\frac{1}{2}|v|^2)$, that corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equals to 0. Furthermore, We notice that

$$\int_{v \cdot n(x) > 0} v \cdot n(x) \sqrt{2\pi} M(v) dv = \int_{v \cdot n(x) < 0} |v \cdot n(x)| \sqrt{2\pi} M(v) dv = 1,$$

which expresses the conservation of mass at the boundary. Here we take the temperature of the wall to be constant and equal to 1.

2.2. Nondimensionalized Form of the Boltzmann Equation. We consider a sequence of renormalized solutions $F_\epsilon(t, x, v)$ to the rescaled Boltzmann equation

$$\begin{aligned} \epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon &= \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \\ F_\epsilon(0, x, v) &= F_\epsilon^{\text{in}}(x, v) \geq 0 \quad \text{on } \mathcal{O}, \\ \gamma_- F_\epsilon &= (1 - \alpha) L \gamma_+ F_\epsilon + \alpha K \gamma_+ F_\epsilon \quad \text{on } \mathbb{R}^+ \times \Sigma_-. \end{aligned} \quad (2.3)$$

The Boltzmann collision operator \mathcal{B} acts only on the v argument of F and is formally given by

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (F'_1 F' - F_1 F) b(\omega, v_1 - v) d\omega dv_1,$$

where v_1 ranges over \mathbb{R}^D endowed with its Lebesgue measure dv_1 , while ω ranges over the unit sphere $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The F'_1, F', F_1 , and F appearing in the integrand designate $F(t, x, \cdot)$ evaluated at the velocities v'_1, v', v_1 and v , respectively, where the primed velocities are defined by

$$v'_1 = v_1 - \omega[\omega \cdot (v_1 - v)], \quad v' = v + \omega[\omega \cdot (v_1 - v)],$$

for any given $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$. This expresses the conservation of momentum and energy for particle pairs after a collision, namely,

$$v + v_1 = v' + v'_1, \quad |v|^2 + |v_1|^2 = |v'|^2 + |v'_1|^2.$$

The collision kernel b is a positive, locally integrable function and has the classical form

$$b(\omega, v) = |v| \Sigma(|\omega \cdot \hat{v}|, |v|),$$

where $\hat{v} = v/|v|$ and Σ is the specific differential cross section. This symmetry implies that the quantity $\int b(\omega, v) d\omega$ is a function of $|v|$ only. The DiPerna-Lions theory requires that b satisfies

$$\lim_{|v| \rightarrow \infty} \frac{1}{1 + |v|^2} \iint_{\mathbb{S}^{D-1} \times K} b(\omega, v_1 - v) d\omega dv_1 = 0 \quad (2.4)$$

for any compact set $K \subset \mathbb{R}^D$. There are some additional assumptions on b needed in [27]. For the convenience of the reader, we list these assumptions here.

A major role will be played by the attenuation coefficient $a(v)$, which is defined as

$$a(v) = \int_{\mathbb{R}^D} \bar{b}(v_1 - v) M_1 dv_1 = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} b(\omega, v_1 - v) d\omega M_1 dv_1.$$

A few facts about $a(v)$ are readily evident from what we have already assumed. Because (2.4) holds, one can show that

$$\lim_{|v| \rightarrow \infty} \frac{a(v)}{1 + |v|^2} = 0. \quad (2.5)$$

Our *second assumption* regarding the collision kernel b is that $a(v)$ satisfies a lower bound of the form

$$C_a(1 + |v|)^\alpha \leq a(v), \quad (2.6)$$

for some constant $C_a > 0$ and $\alpha \in \mathbb{R}$. The *third assumption* is that there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} \right|^s a(v_1) M_1 dv_1 \right)^{\frac{1}{s}} \leq C_b. \quad (2.7)$$

Another major role in what follows will be played by the linearized around the global Maxwellian M collision operator \mathcal{L} , which is defined by

$$\mathcal{L}\tilde{g} = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}'_1) b(\omega, v_1 - v) d\omega M_1 dv_1. \quad (2.8)$$

One has the decomposition

$$\frac{1}{a} \mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+,$$

where the loss operator \mathcal{K}^- and the gain operator \mathcal{K}^+ are defined by

$$\begin{aligned} \mathcal{K}^- \tilde{g} &= \frac{1}{a} \int_{\mathbb{R}^D} \tilde{g}_1 \bar{b}(v_1 - v) M_1 dv_1, \\ \mathcal{K}^+ \tilde{g} &= \frac{1}{a} \int_{\mathbb{R}^D} (\tilde{g}' + \tilde{g}'_1) b(\omega, v_1 - v) d\omega M_1 dv_1. \end{aligned}$$

The *fourth assumption* regarding the collision kernel b is that

$$\mathcal{K}^+ : L^2(aMdv) \rightarrow L^2(aMdv) \text{ is compact.} \quad (2.9)$$

Combining the gain operator assumption (2.9) and the loss operator assumption (2.7), we conclude that

$$\frac{1}{a} \mathcal{L} : L^p(aMdv) \rightarrow L^p(aMdv) \text{ is Fredholm}$$

for every $p \in (1, \infty)$. From this Fredholm property we can define the psuedo-inverse of \mathcal{L} , called \mathcal{L}^{-1} :

$$\mathcal{L}^{-1} : L^p(a^{1-p}Mdv) \cap \text{Null}^\perp(\mathcal{L}) \rightarrow L^p(aMdv).$$

Moreover, \mathcal{L}^{-1} is a bounded operator.

The *fifth assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that \bar{b} satisfies

$$\frac{\bar{b}(v_1 - v)}{1 + \delta \frac{\bar{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leq C_\delta (1 + a(v_1))(1 + a(v)) \quad \text{for every } v_1, v \in \mathbb{R}^D. \quad (2.10)$$

It is well known that the null space of the linearized Boltzmann operator \mathcal{L} is given by $\text{Null}(\mathcal{L}) \equiv \text{span}\{1, v_1, \dots, v_D, |v|^2\}$. Let \mathcal{P} be the orthogonal projection from $L^2(Mdv)$ onto $\text{Null}(\mathcal{L})$, namely,

$$\mathcal{P}\tilde{g} = \langle \tilde{g} \rangle + v \cdot \langle \tilde{g} \rangle + \left(\frac{|v|^2}{2} - \frac{D}{2}\right) \langle \left(\frac{1}{D}|v|^2 - 1\right) \tilde{g} \rangle, \quad (2.11)$$

where the notation $\langle \cdot \rangle$ is defined below in (2.16). Furthermore, we define $\mathcal{P}^\perp = \mathcal{I} - \mathcal{P}$. The matrix-valued function $A(v)$ and the vector-valued function $B(v)$ are defined by

$$A(v) = v \otimes v - \frac{1}{D}|v|^2 I, \quad B(v) = \frac{1}{2}|v|^2 v - \frac{D+2}{2}v. \quad (2.12)$$

We also define a scalar-valued function $C(v)$ by

$$C(v) = \frac{1}{4}|v|^4 - \frac{D+2}{2}|v|^2 + \frac{D(D+2)}{4}. \quad (2.13)$$

It is easy to see that each entry of A , B and C are in $L^2(a^{-1}Mdv) \cap \text{Null}^\perp(\mathcal{L})$. Furthermore, C is perpendicular to each entry of A and B . We also introduce $\hat{A} \in L^2(aMdv; \mathbb{R}^{D \times D})$ and $\hat{B} \in L^2(aMdv; \mathbb{R}^D)$ by

$$\hat{A} = \mathcal{L}^{-1}A, \quad \hat{B} = \mathcal{L}^{-1}B. \quad (2.14)$$

Next, for the sake of simplicity, we take the following normalizations:

$$\begin{aligned} \iiint_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D} b(\omega, v_1 - v) d\omega M_1 dv_1 dv &= 1, \\ \int_{\mathbb{S}^{D-1}} d\omega &= 1, \quad \int_{\mathbb{R}^D} Mdv = 1, \quad \int_{\Omega} dx = 1, \end{aligned}$$

associated with the domains $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, \mathbb{S}^{D-1} , \mathbb{R}^D and Ω respectively, and

$$\iint_{\Omega \times \mathbb{R}^D} F_\varepsilon^{\text{in}} dx dv = 1, \quad (2.15)$$

associated with the initial data $F_\varepsilon^{\text{in}}$.

Because Mdv is a positive unit measure on \mathbb{R}^D , we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v) Mdv, \quad (2.16)$$

and the inner product on $L^2(Mdv)$

$$\langle \xi, \eta \rangle = \int_{\mathbb{R}^D} \xi(v) \overline{\eta(v)} Mdv,$$

where $\bar{\eta}$ denotes the complex conjugate of η . Moreover, we also use the following average on the boundary

$$\langle \xi \rangle_{\partial\Omega} = \int_{\mathbb{R}^D} \xi(v) [\mathbf{n}(x) \cdot v] \sqrt{2\pi} Mdv, \quad (2.17)$$

from which we have $\langle \mathbf{1}_{\Sigma_+} \rangle_{\partial\Omega} = -\langle \mathbf{1}_{\Sigma_-} \rangle_{\partial\Omega} = 1$. Because $d\mu = b(\omega, v_1 - v) d\omega M_1 dv_1 M dv$ is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\langle \Xi \rangle\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$

$$\langle\langle \Xi \rangle\rangle = \iiint_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D} \Xi(\omega, v_1, v) d\mu.$$

The measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1), \quad (\omega, v_1, v) \mapsto (\omega, v'_1, v').$$

These are called *dμ-symmetries*.

2.3. Navier-Stokes Scaling. The incompressible NSF system can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic densities F_ε about the absolute Maxwellian M are scaled to be of order ε . More precisely, we take

$$F_\varepsilon = MG_\varepsilon = M(1 + \varepsilon g_\varepsilon). \quad (2.18)$$

Rewriting equation (2.3) for G_ε yields

$$\begin{aligned} \varepsilon \partial_t G_\varepsilon + v \cdot \nabla_x G_\varepsilon &= \frac{1}{\varepsilon} \mathcal{Q}(G_\varepsilon, G_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \\ G_\varepsilon(0, x, v) &= G_\varepsilon^{\text{in}}(x, v) \quad \text{on } \mathcal{O}, \\ \gamma_- G_\varepsilon &= (1 - \alpha)L\gamma_+ G_\varepsilon + \alpha \langle \gamma_+ G_\varepsilon \rangle_{\partial\Omega} \quad \text{on } \mathbb{R}^+ \times \Sigma_-, \end{aligned} \quad (2.19)$$

where the collision kernel \mathcal{Q} is now given by

$$\mathcal{Q}(G, G) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G'_1 G' - G_1 G) b(\omega, v_1 - v) d\omega M_1 dv_1.$$

In terms of g_ε the system (2.3) finally reads

$$\begin{aligned} \varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon &= \mathcal{Q}(g_\varepsilon, g_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \\ g_\varepsilon(0, x, v) &= g_\varepsilon^{\text{in}}(x, v) \quad \text{on } \mathcal{O}, \\ \gamma_- g_\varepsilon &= (1 - \alpha)L\gamma_+ g_\varepsilon + \alpha \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega} \quad \text{on } \mathbb{R}^+ \times \Sigma_-. \end{aligned} \quad (2.20)$$

2.4. A Priori Estimates. Due to the presence of the boundary, the classical *a priori* estimates for the Boltzmann equation, namely the entropy and energy bounds, are modified. First, because all particles arriving at the boundary are reflected or diffused, we have conservation of mass, which can be written as

$$\int_{\Omega} \langle G_\varepsilon \rangle dx = \int_{\Omega} \langle G_\varepsilon^{\text{in}} \rangle dx = 1.$$

Multiplying the equation (2.19) by $\log(G_\varepsilon)$ and integrating in x and v , we get formally

$$\begin{aligned} &\varepsilon \partial_t \int_{\Omega} \langle G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1 \rangle dx + \iint_{\Sigma} (G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1) v \cdot n(x) d\sigma_x M dv \\ &= \frac{1}{\varepsilon} \int_{\Omega} \langle \log(G_\varepsilon) \mathcal{Q}(G_\varepsilon, G_\varepsilon) \rangle dx. \end{aligned}$$

By denoting $h(z) = (1+z)\log(1+z) - z$, for $z > -1$ and using that it is a convex function, we can compute the boundary term in the following way:

$$\begin{aligned}\tilde{\mathcal{E}}_\varepsilon(\gamma_+ G_\varepsilon) &= \iint_{\Sigma} [G_\varepsilon \log(G_\varepsilon) - G_\varepsilon + 1] v \cdot n(x) d\sigma_x M dv \\ &= \iint_{\Sigma_+} [h(\varepsilon \gamma_+ g_\varepsilon) - h((1 - \alpha_\varepsilon)\varepsilon \gamma_+ g_\varepsilon + \alpha_\varepsilon \langle \varepsilon \gamma_+ g_\varepsilon \rangle_{\partial\Omega})] v \cdot n(x) M dv d\sigma_x \\ &\geq \iint_{\Sigma_+} [h(\varepsilon \gamma_+ g_\varepsilon) - (1 - \alpha_\varepsilon)h(\varepsilon \gamma_+ g_\varepsilon) - \alpha_\varepsilon h(\langle \varepsilon \gamma_+ g_\varepsilon \rangle_{\partial\Omega})] v \cdot n(x) M dv d\sigma_x \\ &= \frac{\alpha_\varepsilon}{\sqrt{2\pi}} \mathcal{E}(\gamma_+ G_\varepsilon),\end{aligned}$$

where $\mathcal{E}(\gamma_+ G_\varepsilon)$, the so-called Darrozès-Guiraud information, is given by

$$\mathcal{E}(\gamma_+ G_\varepsilon) = \int_{\partial\Omega} [\langle h(\varepsilon \gamma_+ g_\varepsilon) \rangle_{\partial\Omega} - h(\varepsilon \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega})] d\sigma_x.$$

Jensen's inequality implies that $\mathcal{E}(\gamma_+ G_\varepsilon) \geq 0$. Noticing that $\tilde{\mathcal{E}}_\varepsilon(\gamma_+ G_\varepsilon) \geq \frac{\alpha_\varepsilon}{\sqrt{2\pi}} \mathcal{E}(\gamma_+ G_\varepsilon)$, we get the entropy inequality

$$H(G_\varepsilon(t)) + \int_0^t \left(\frac{1}{\varepsilon^2} R(G_\varepsilon(s)) + \frac{1}{\varepsilon} \tilde{\mathcal{E}}_\varepsilon(\gamma_+ G_\varepsilon(s)) \right) ds \leq H(G_\varepsilon^{\text{in}}), \quad (2.21)$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\Omega} \langle G \log(G) - G + 1 \rangle dx,$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\Omega} \left\langle \left\langle \frac{1}{4} \log \left(\frac{G'_1 G'}{G_1 G} \right) (G'_1 G' - G_1 G) \right\rangle \right\rangle dx.$$

2.5. DiPerna-Lions-(Mischler) Solutions. We will work in the setting of renormalized solutions which were initially constructed by DiPerna and Lions [12] over the whole space \mathbb{R}^D for any initial data satisfying natural physical bounds. Recently, their result was extended to the case of a bounded domain by Mischler [35, 36, 37] with general Maxwell boundary conditions (2.1).

The DiPerna-Lions-(Mishler) theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial value problems that are obtained by multiplying (2.19) by $\Gamma'(G_\varepsilon)$:

$$\begin{aligned}(\varepsilon \partial_t + v \cdot \nabla_x) \Gamma(G_\varepsilon) &= \frac{1}{\varepsilon} \Gamma'(G_\varepsilon) \mathcal{Q}(G_\varepsilon, G_\varepsilon) \quad \text{on } \mathbb{R}^+ \times \mathcal{O}, \\ G_\varepsilon(0, \cdot, \cdot) &= G_\varepsilon^{\text{in}} \geq 0 \quad \text{on } \mathcal{O}.\end{aligned} \quad (2.22)$$

Here the admissible function $\Gamma : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and for some constant $C_\Gamma < \infty$ its derivative satisfies

$$|\Gamma'(z)| \sqrt{1+z} \leq C_\Gamma. \quad (2.23)$$

The weak formulation of the renormalized Boltzmann equation (2.22) is given by

$$\begin{aligned} & \varepsilon \int_{\Omega} \langle \Gamma(G_{\varepsilon}(t_2))Y \rangle dx - \varepsilon \int_{\Omega} \langle \Gamma(G_{\varepsilon}(t_1))Y \rangle dx \\ & - \int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma(G_{\varepsilon})v \cdot \nabla_x Y \rangle dx dt + \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \Gamma(\gamma G_{\varepsilon})Y[n(x) \cdot v] \rangle d\sigma_x dt \\ & = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle \Gamma'(G_{\varepsilon})\mathcal{Q}(G_{\varepsilon}, G_{\varepsilon})Y \rangle dx dt, \end{aligned} \quad (2.24)$$

for every $Y \in C^1 \cap L^\infty(\bar{\Omega} \times \mathbb{R}^D)$ and every $[t_1, t_2] \subset [0, \infty]$. Moreover, the boundary condition is also understood in the renormalized sense:

$$\Gamma(\gamma_- G_{\varepsilon}) = \Gamma\left((1 - \alpha)L\gamma_+ G_{\varepsilon} + \alpha \widetilde{F}_{\varepsilon}\right) \quad \text{on } \mathbb{R}^+ \times \Sigma_-, \quad (2.25)$$

where the equality holds almost everywhere and in the sense of distribution.

Proposition 2.1. (Renormalized solutions in bounded domain [37]) *Let b satisfy the condition (2.4). Given any initial data $G_{\varepsilon}^{\text{in}}$ satisfying*

$$\iint_{\mathcal{O}} G_{\varepsilon}^{\text{in}}(1 + |v|^2 + |\log G_{\varepsilon}^{\text{in}}|) M dv dx < +\infty, \quad (2.26)$$

there exists at least one $G_{\varepsilon} \geq 0$ in $C([0, \infty); L^1(M dv dx))$ such that (2.24) and (2.25) hold for all admissible functions Γ . Moreover, G_{ε} satisfies the following global entropy inequality for all $t > 0$:

$$H(G_{\varepsilon}(t)) + \frac{1}{\varepsilon^2} \int_0^t R(G_{\varepsilon}(s)) ds + \frac{1}{\varepsilon} \int_0^t \widetilde{\mathcal{E}}_{\varepsilon}(\gamma_+ G_{\varepsilon}(s)) \leq H(G_{\varepsilon}^{\text{in}}). \quad (2.27)$$

3. STATEMENT OF THE MAIN RESULTS

In this section we state our main results on justifying the incompressible NSF limits with different boundary conditions depending on the quotient between the accommodation coefficients α_{ε} and the Knudsen number ε .

3.1. Dirichlet Boundary Condition. The main theorem of this paper is the following strong convergence to the NSF system with Dirichlet boundary condition when the accommodation coefficient α_{ε} is much larger than the Knudsen number ε , i.e. $\frac{\alpha_{\varepsilon}}{\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Theorem 3.1. (Dirichlet Boundary Condition) *Let b be a collision kernel that satisfies conditions (2.5), (2.6), (2.7), (2.9) and (2.10). Let $G_{\varepsilon}^{\text{in}}$ be any family of non-negative measurable functions of (x, v) satisfying (2.26) and the renormalization (2.15). Let $g_{\varepsilon}^{\text{in}}$ be the associated family of fluctuations given by $G_{\varepsilon}^{\text{in}} = 1 + \varepsilon g_{\varepsilon}^{\text{in}}$. Assume that the families $G_{\varepsilon}^{\text{in}}$ and $g_{\varepsilon}^{\text{in}}$ satisfy*

$$H(G_{\varepsilon}^{\text{in}}) \leq C^{\text{in}} \varepsilon^2, \quad (3.1)$$

and

$$\lim_{\varepsilon \rightarrow 0} \left(\langle g_{\varepsilon}^{\text{in}} \rangle, \langle v g_{\varepsilon}^{\text{in}} \rangle, \left\langle \left(\frac{|v|^2}{D} - 1 \right) g_{\varepsilon}^{\text{in}} \right\rangle \right) = (\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}), \quad (3.2)$$

in the sense of distributions for some $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(dx; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$. Let G_{ε} be any family of DiPerna-Lions renormalized solutions to the Boltzmann equation (2.19) that have $G_{\varepsilon}^{\text{in}}$ as initial values, and the accommodation coefficient α_{ε} satisfies

$$\alpha_{\varepsilon} = \sqrt{2\pi} \chi \sqrt{\varepsilon}. \quad (3.3)$$

Then the family of fluctuations g_ε given by (2.18) is relatively compact in $L^1_{loc}(dt; L^1(\sigma M dv dx))$. Every limit point g of g_ε has the infinitesimal Maxwellian form

$$g = v \cdot u + \left(\frac{1}{2}|v|^2 - \frac{D+2}{2}\right) \theta, \quad (3.4)$$

where $(u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R})) \cap L^2(dt; H^1(dx; \mathbb{R}^D \times \mathbb{R}))$ with mean zero over Ω , and it satisfies the NSF system with Dirichlet boundary condition (1.1), (1.2), and (1.3), where kinematic viscosity ν and thermal conductivity κ are given by

$$\nu = \frac{1}{(D-1)(D+2)} \langle \hat{A} : \mathcal{L} \hat{A} \rangle, \quad \kappa = \frac{1}{D} \langle \hat{B} \cdot \mathcal{L} \hat{B} \rangle. \quad (3.5)$$

The initial data is given by

$$u^0 = \mathbb{P} u^{\text{in}}, \quad \theta^0 = \frac{D}{D+2} \theta^{\text{in}} - \frac{2}{D+2} \rho^{\text{in}}. \quad (3.6)$$

Here the operator \mathbb{P} is the Leray's projection on the space of divergence free vector fields. Moreover, every subsequence g_{ε_k} of g_ε that converges to g as $\varepsilon_k \rightarrow 0$ also satisfies

$$\begin{aligned} \langle v g_{\varepsilon_k} \rangle &\rightarrow u \quad \text{in } L^p_{loc}(dt; L^1(dx; \mathbb{R}^D)), \\ \langle \left(\frac{1}{D}|v|^2 - 1\right) g_{\varepsilon_k} \rangle &\rightarrow \theta \quad \text{in } L^p_{loc}(dt; L^1(dx; \mathbb{R})) \quad \text{for every } 1 \leq p < \infty. \end{aligned} \quad (3.7)$$

Furthermore, $\frac{1}{\varepsilon} \mathcal{P}^\perp g_\varepsilon$ is relatively compact in $w-L^1_{loc}(dt; w-L^1(\sigma M dv dx))$. For every subsequence ε_k so that g_{ε_k} converges to g ,

$$\begin{aligned} \frac{1}{\varepsilon} \mathcal{P}^\perp g_{\varepsilon_k} &\rightarrow \frac{1}{2} A : u \otimes u + B \cdot u \theta + \frac{1}{2} C \theta^2 \\ &\quad - \hat{A} : \nabla_x u - \hat{B} \cdot \nabla_x \theta, \quad \text{in } w-L^1_{loc}(dt; w-L^1(\sigma M dv dx)), \end{aligned} \quad (3.8)$$

as $\varepsilon_k \rightarrow 0$, where A, B, C and \hat{A}, \hat{B} are defined in (2.12), (2.13) and (2.14).

Remark: In the formal Chapman-Enskog expansion,

$$g_\varepsilon = g + \varepsilon \mathcal{P}^\perp g_1 + \varepsilon \mathcal{P} g_1 + \varepsilon^2 g_2 + \dots,$$

where g is given by (3.4) and $\mathcal{P}^\perp g_1$ is the righthand side term in (3.8). In previous works [19, 20, 27], under the assumptions (3.1) and (3.2), the convergence to (3.4) and (3.7) are only in $w-L^1$. So the convergence to the quadratic term (3.8), which is the first correction to the infinitesimal Maxwellian that one obtains from the Chapman-Enskog expansion with the Navier-Stokes scaling, could not be obtained. In Theorem 3.1, by showing the acoustic waves are instantaneously damped, we justify not only the strong convergence to the leading order term g , but also weak convergence to the kinetic part of the next order corrector (3.8).

3.2. Navier Boundary Condition. The second result is about Navier boundary condition. For this case, although the coupled viscous boundary layer and the Knudsen layer still have dissipative effect, however, the damping happens a longer time scale $O(1)$. Consequently, unlike the Dirichlet boundary condition case, the fast acoustic waves can be damped, but *not instantaneously*. Nevertheless, we can show the weak convergence result, thus justify the NSF limit with slip Navier boundary condition, while the linear Stokes-Fourier limit was justified in [33].

Theorem 3.2. (Navier Boundary Condition) *With the same assumptions with Theorem 3.1, except that the accommodation coefficients satisfy*

$$\frac{\alpha_\varepsilon}{\sqrt{2\pi\varepsilon}} \rightarrow \chi, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.9)$$

Then the family g_ε is relatively compact in $w-L^1_{loc}(dt; w-L^1(\sigma M dv dx))$. Every limit point g of g_ε in $w-L^1_{loc}(dt; w-L^1(\sigma M dv dx))$ has the infinitesimal Maxwellian form as (3.4) in which $(u, \theta) \in C([0, \infty); L^2(dx; \mathbb{R}^D \times \mathbb{R})) \cap L^2(dt; H^1(dx; \mathbb{R}^D \times \mathbb{R}))$ is a Larey solution of the NSF system with Navier boundary condition (1.1), (1.2), and (1.4), where kinematic viscosity ν and thermal conductivity κ are given by (3.5), the initial data is given by (3.6).

Moreover, every subsequence g_{ε_k} of g_ε that converges to g as $\varepsilon_k \rightarrow 0$ also satisfies

$$\begin{aligned} \mathbb{P}\langle v g_{\varepsilon_k} \rangle &\rightarrow \mathbf{u} \quad \text{in } C([0, \infty); \mathcal{D}'(\Omega; \mathbb{R}^D)), \\ \langle (\frac{1}{D+2}|v|^2 - 1)g_{\varepsilon_k} \rangle &\rightarrow \theta \quad \text{in } C([0, \infty); w\text{-}L^1(\Omega; \mathbb{R})). \end{aligned} \quad (3.10)$$

Remark: For the Navier-slip boundary condition case, since the convergence is weak, the convergence (3.8), i.e. the justification of the first correction to the infinitesimal Maxwellian in the Chapman-Enskog expansion can not be obtained.

4. GEOMETRY OF THE BOUNDARY $\partial\Omega$

In this section, we collect some differential geometry properties related to the boundary $\partial\Omega$ which can be considered as a $(D-1)$ dimension Riemannian manifold with a metric induced from the standard Euclidian metric of \mathbb{R}^D . From the following classical result in geometry (for the proof, see [42]), there is a tubular neighborhood $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$ of $\partial\Omega$ such that the nearest point projection map is well defined.

Lemma 4.1. *If $\partial\Omega$ is a compact C^k submanifold of dimension $D-1$ embedded in \mathbb{R}^D , then there is $\delta = \delta_{\partial\Omega} > 0$ and a map $\pi \in C^{k-1}(\Omega_\delta; \mathbb{R}^D)$ such that the following properties hold:*

(i): *for all $x \in \Omega \subset \mathbb{R}^D$ with $\text{dist}(x, \partial\Omega) < \delta$;*

$$\pi(x) \in \partial\Omega, \quad x - \pi(x) \in T_{\Pi(x)}^\perp(\partial\Omega), \quad |x - \pi(x)| = \text{dist}(x, \partial\Omega), \text{ and}$$

$$|z - x| > \text{dist}(x, \partial\Omega) \quad \text{for any } z \in \partial\Omega \setminus \{\pi(x)\};$$

(ii):

$$\pi(x+z) \equiv x, \quad \text{for } x \in \partial\Omega, z \in T_x(\partial\Omega)^\perp, |z| < \delta,$$

(iii): *Let $\text{Hess}\Pi^x$ denote the Hessian of π at x , then*

$$\text{Hess}\pi^x(V_1, V_2) = h_x(V_1, V_2), \quad \text{for } x \in \partial\Omega \quad V_1, V_2 \in T_x(\partial\Omega),$$

where h_x is the second fundamental form of $\partial\Omega$ at x .

The viscous boundary layer has significantly different behavior over the tangential and normal directions near the boundary. This inspire us to consider the following new coordinate system, which we call the curvilinear coordinate for the tubular neighborhood Ω_δ defined in Lemma 4.1. Because $\partial\Omega$ is a $(D-1)$ dimensional manifold, so locally $\pi(x)$ can be represented as

$$\pi(x) = (\pi^1(x), \dots, \pi^{D-1}(x)). \quad (4.1)$$

More precisely, the representation (4.1) could be understood in the following sense: we can introduce a new coordinate system (ξ^1, \dots, ξ^D) by a homeomorphism which locally defined as $\xi : \xi(x) = (\xi'(x), \xi^D(x))$ where $\xi' = (\xi^1, \dots, \xi^{D-1})$, such that $\xi(\pi(x)) = (\xi', 0)$ and $d(x) = \xi^D$, where $d(x)$ is the distance function to the boundary $\partial\Omega$, i.e.

$$d(x) = \text{dist}(x, \partial\Omega) = |x - \pi(x)|. \quad (4.2)$$

For the simplicity of notation, we denote “ $\xi'(x) = \pi(x)$ ” which is the meaning of (4.1).

It is easy to see that $\nabla_x d$ is perpendicular to the level surface of the distance function d , i.e. the set $S^z = \{x \in \Omega : d(x) = z\}$. In particular, on the boundary, $\nabla_x d$ is perpendicular to $S_0 = \partial\Omega$. Without loss of generality, we can normalize the distance function so that $\nabla_x d(x) = -n(x)$ when $x \in \partial\Omega$. By the definition of the projection Π , we have

$$\pi(x + t\nabla_x d(x)) = \pi(x) \quad \text{for } t \text{ small}, \quad (4.3)$$

and consequently, $\nabla_x \pi^\alpha \cdot \nabla_x d = 0$, for $\alpha = 1, \dots, D-1$. In particular, for t small enough, $\nabla_x \pi^\alpha(x) \in T_x(\partial\Omega)$ when $x \in \partial\Omega$.

Next, we calculate the induced Riemannian metric from \mathbb{R}^D on $\partial\Omega$. In a local coordinate system, these Riemannian metric can be represented as

$$g = g_{\alpha\beta} d\pi^\alpha \otimes d\pi^\beta,$$

where $g_{\alpha\beta} = \langle \frac{\partial}{\partial \pi^\alpha}, \frac{\partial}{\partial \pi^\beta} \rangle$. Noticing that $\frac{\partial}{\partial x^i} = \frac{\partial \pi^\alpha}{\partial x^i} \frac{\partial}{\partial \pi^\alpha}$, and $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \delta_{ij}$, the metric $g_{\alpha\beta}$ can be determined by

$$g_{\alpha\beta} \frac{\partial \pi^\alpha}{\partial x^i} \frac{\partial \pi^\beta}{\partial x^i} = 1.$$

5. ACOUSTIC MODES

5.1. Acoustic Operator \mathcal{A} . Recall that the Leray's projection \mathbb{P} on the space of divergence-free vector fields and \mathbb{Q} on the space of gradients are defined by

$$\mathbb{P} = \mathbb{I} - \mathbb{Q},$$

where $\mathbb{Q}u = \nabla_x q$ and q solves

$$\begin{aligned} \Delta_x q &= \nabla_x \cdot u \quad \text{in } \Omega, \\ \nabla_x q \cdot n &= u \cdot n \quad \text{on } \partial\Omega, \quad \text{and} \quad \int_{\Omega} q \, dx = 0. \end{aligned} \tag{5.1}$$

We define Hilbert spaces

$$\begin{aligned} \mathbb{H} &= \{U = (\rho, u, \theta) \in L^2(dx; \mathbb{C} \times \mathbb{C}^D \times \mathbb{C})\}, \\ \mathbb{V} &= \left\{ U \in \mathbb{H} : \int_{\Omega} |\nabla_x U|^2 \, dx < \infty \right\}, \end{aligned}$$

endowed with inner product

$$\langle U_1, U_2 \rangle_{\mathbb{H}} = \int_{\Omega} (\rho_1 \bar{\rho}_2 + u_1 \cdot \bar{u}_2 + \frac{D}{2} \theta_1 \bar{\theta}_2) \, dx, \tag{5.2}$$

where \bar{f} denotes the complex conjugate of the complex-valued function f . Next, we define the acoustic operator \mathcal{A} :

$$\mathcal{A} \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_x \cdot u \\ \nabla_x (\rho + \theta) \\ \frac{2}{D} \nabla_x \cdot u \end{pmatrix}, \tag{5.3}$$

over the domain

$$\text{Dom}(\mathcal{A}) = \{U = (\rho, u, \theta) \in \mathbb{V} : u \cdot n = 0 \text{ on } \partial\Omega\}.$$

The null space of \mathcal{A} and its orthogonal with respect to the inner product (5.2) are characterized as

$$\text{Null}(\mathcal{A}) = \{(-\varphi, w, \varphi) \in \mathbb{V} : \nabla_x \cdot w = 0 \quad \text{and} \quad w \cdot n = 0 \quad \text{on } \partial\Omega\}, \tag{5.4}$$

and

$$\text{Null}(\mathcal{A})^\perp = \{(\rho, u, \theta) \in \mathbb{V} : \theta = \frac{2}{D} \rho, u = \nabla_x \phi, \text{ for some } \phi \in H^1(\Omega)\}, \tag{5.5}$$

respectively. Because $\text{Null}(\mathcal{A})$ includes the incompressibility and Boussinesq relations, we call it *incompressible* regime. We will see in the next subsection that $\text{Null}(\mathcal{A})^\perp$ is spanned by the eigenspaces of the acoustic operator \mathcal{A} , so we call it *acoustic* regime.

For any $U = (\rho, u, \theta) \in \mathbb{H}$, we can define Π and Π^\perp the projections to the incompressible regime $\text{Null}(\mathcal{A})$ and acoustic regime $\text{Null}(\mathcal{A})^\perp$ respectively as follows:

$$\begin{aligned} \Pi U &= \left(\frac{2}{D+2} \rho - \frac{D}{D+2} \theta, \mathbb{P}u, \frac{D}{D+2} \theta - \frac{2}{D+2} \rho \right), \\ \Pi^\perp U &= \left(\frac{D}{D+2} (\rho + \theta), \mathbb{Q}u, \frac{2}{D+2} (\rho + \theta) \right). \end{aligned}$$

5.2. Eigenspaces of \mathcal{A} . The eigenvalues and eigenvectors of the acoustic operator \mathcal{A} in a bounded domain can be constructed from those of the Laplace operator with Neumann boundary condition in the following way: Let $\frac{D}{D+2}[\lambda^k]^2, \lambda^k > 0, k \in \mathbb{N}$ be the nondecreasing sequence of eigenvalues of the Laplace operator $-\Delta_N$ with homogeneous Neumann boundary condition, and Ψ^k be the corresponding orthonormal basis of $L^2(\Omega)$ eigenfunctions:

$$-\Delta_x \Psi^k = \frac{D}{D+2}[\lambda^k]^2 \Psi^k \quad \text{in } \Omega, \quad \nabla_x \Psi^k \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (5.6)$$

More specifically,

$$0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^k \rightarrow +\infty, \quad \text{as } k \rightarrow \infty.$$

Let τ denote either $+$ or $-$, and $\lambda^{\tau,k} = \tau\lambda^k$. It can be verified that $i\lambda^{\tau,k}$ are non-zero eigenvalues of \mathcal{A} and

$$U^{\tau,k} = \sqrt{\frac{D+2}{2D}} \left(\frac{D}{D+2} \Psi^k, \frac{\nabla_x \Psi^k}{i\lambda^{\tau,k}}, \frac{2}{D+2} \Psi^k \right)^\top \quad (5.7)$$

are the corresponding normalized eigenvectors, i.e.

$$\mathcal{A}U^{\tau,k} = i\lambda^{\tau,k}U^{\tau,k}, \quad (5.8)$$

and furthermore, $U^{\tau,k}$ span $\text{Null}(\mathcal{A})^\perp$ under the inner product (5.2). Consequently we have an orthonormal basis of the acoustic modes, i.e.

$$\text{Null}(\mathcal{A})^\perp = \overline{\text{Span} \{U^{\tau,k} | k \in \mathbb{N}, \tau = \pm\}}^{L^2}.$$

Moreover, we can use the components of $U^{\tau,k}$ to construct the infinitesimal Maxwellians $g^{\tau,k}$ which are in the null space of \mathcal{L} :

$$g^{\tau,k} = \sqrt{\frac{D+2}{2D}} \left\{ \frac{D}{D+2} \Psi^k + v \cdot \frac{\nabla_x \Psi^k}{i\lambda^{\tau,k}} + \frac{2}{D+2} \Psi^k \left(\frac{|v|^2}{2} - \frac{D}{2} \right) \right\}. \quad (5.9)$$

These infinitesimal Maxwellians will be the building blocks of the approximate eigenfunctions of $\frac{1}{\varepsilon}\mathcal{L} - v \cdot \nabla_x$.

5.3. Conditions on Ψ^k . Note that $\Psi^k, k \geq 1$ are solutions to the Neumann boundary condition equation (5.6), so some *orthogonality condition* is required for the eigenfunctions associated to the eigenvalues with multiplicity greater than 1. Assume that λ^2 is an eigenvalue of (5.6) and denote by $H_0 = H_0(\lambda)$ the eigenspace associated to λ^2 , i.e.

$$H_0(\lambda) = \{\Psi \in \text{Dom}(-\Delta_x) : -\Delta_x \Psi = \lambda^2 \Psi \text{ in } \Omega, \frac{\partial \Psi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\} \quad (5.10)$$

where $\text{Dom}(-\Delta_x) = H^2(\Omega) \cap \{\Psi | \frac{\partial \Psi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ denotes the domain of $-\Delta_x$ with Neumann boundary condition. On the finite dimensional space $H_0(\lambda)$, we can define a quadratic form Q_1 . Its associated bilinear form that we still denote Q_1 and a symmetric operator $L_1 = L_1^\lambda$ by

$$Q_1(\Psi, \Phi) = \int_\Omega L_1(\Psi) \Phi \, dx = \int_\Omega L_1(\Phi) \Psi \, dx. \quad (5.11)$$

The eigenspace $H_0(\lambda)$ is endowed with an orthogonality condition

$$Q_1(\Psi^k, \Psi^l) = 0, \quad \text{if } \Psi^k, \Psi^l \in H_0(\lambda) \text{ and } k \neq l. \quad (5.12)$$

This condition means that the eigenvectors Ψ^k for $\lambda^k = \lambda$ are orthogonal for the symmetric operator L_1^λ . Of course, since $L^2(\Omega)$ is the direct sum of the spaces $H_0(\lambda)$ for different λ 's. From the definition of L_1^λ on each eigenspace $H_0(\lambda)$, we can define an operator L_1 on $L^2(\Omega)$ which leaves each eigenspace $H_0(\lambda)$ invariant. But this is not necessary, so we will think of $L_1 = L_1^\lambda$ as acting on $H_0(\lambda)$ for a fixed multiple eigenvalue λ .

The orthogonality condition (5.12) turns out to be enough for the construction of the boundary layer if the eigenvalues of L_1 are simple, namely, if $\lambda_1^k \neq \lambda_1^l$ for all $k \neq l$ such that $\lambda_0^k = \lambda_0^l = \lambda$.

However, if λ_1 is an eigenvalue of L_1 with multiplicity greater than or equal to 2, then we need an extra orthogonality condition. Let $H_1 = H_1(\lambda_1)$ be defined by

$$H_1 = \{\Psi \in H_0 : L_1 \Psi = \lambda_1 \Psi\}. \quad (5.13)$$

On the finite dimensional space H_1 , there exists a quadratic form Q_2 and a symmetric operator L_2 (see the definition below), the extra condition is

$$Q_2(\Psi^k, \Psi^l) = 0, \quad \text{if } \Psi^k, \Psi^l \in H_1(\lambda) \quad \text{and} \quad k \neq l. \quad (5.14)$$

This condition is enough if L_2 has only simple eigenvalue on the vector space H_1 . This process can be continued inductively.

Let us now explain more precisely the condition we have to impose on the eigenvectors of $-\Delta_x$. We can construct recursively, on each eigenspace $H_0(\lambda)$ of $-\Delta_x$, a sequence of symmetric operators $L_q, q \in \mathbb{N}$ in the following way: Let $L_0 = -\Delta_x$, we define L_1 on each one of the eigenspace $H_0(\lambda)$ of L_0 by (5.11). Assume that the operators L_p were constructed for $p \leq q-1, q \geq 2$ in such a way that each operator L_p leaves invariant the eigenspaces of the operators $L_{p'}$ for $p' < p$. Now, to construct L_q , it is enough to construct L_q on each eigenspace $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{q-1}(\lambda_{q-1})$, where $\lambda_1, \lambda_2, \dots, \lambda_{q-1}$ are eigenvalues of L_1, L_2, \dots, L_{q-1} respectively. This is done by constructing a quadratic form Q_q on each space $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{q-1}(\lambda_{q-1})$ and defining L_q by

$$Q_q(\Psi, \Phi) = \int_{\Omega} L_q(\Psi) \Phi \, dx, \quad \text{for all } \Psi, \Phi \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{q-1}(\lambda_{q-1}).$$

The precise construction of the quadratic form Q_q on the space $H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{q-1}(\lambda_{q-1})$ will be done in the proof.

Let $N \in \mathbb{N}$ be an integer. This is the integer that will appear in the order of the approximation in the Proposition 7.1. The eigenvectors Ψ^k for $\lambda_0^k = \lambda$ should be chosen in such a way that they are eigenvectors for all the operators L_n at least for $n \leq N+2$. This implies that they are orthogonal to all the operators L_n for $n \leq N+2$, which means that

$$Q_n(\Psi^k, \Psi^l) = \int_{\Omega} L_n(\Psi^k) \Psi^l = 0, \quad (5.15)$$

if $\Psi^k, \Psi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{n-1}(\lambda_{n-1})$ and $k \neq l$.

Remark: The precise construction of the quadratic form Q_q will be done in the proof of Proposition 7.1.

5.4. The operator $\mathcal{A} - i\lambda^{\tau,k}$. Later on, in the construction of the boundary layers, for each acoustic mode $k \geq 1$ and $\tau = +$ or $-$, we will frequently solve the following linear hyperbolic system for $V^{\tau,k} = (\rho^{\tau,k}, v^{\tau,k}, \theta^{\tau,k})^T$:

$$\begin{aligned} (\mathcal{A} - i\lambda^{\tau,k}) V^{\tau,k} &= i\mu^{\tau,k} U^{\tau,k} + F^{\tau,k}, \\ v^{\tau,k} \cdot n &= g^{\tau,k} \quad \text{on } \partial\Omega. \end{aligned} \quad (5.16)$$

where $\mu^{\tau,k}$, $F^{\tau,k}$ and $g^{\tau,k}$ are given, and $U^{\tau,k}$ is defined in (5.7).

Remark: Strictly speaking, (5.16) is not rigorous because $v^{\tau,k} \cdot n$ is non-zero, so $V^{\tau,k}$ is not in the domain of \mathcal{A} . For notational simplicity, we still use \mathcal{A} in (5.16) and later on, just mean the expression of \mathcal{A} in (5.3) regardless of the domain.

To solve the system (5.16), the main difficulty is that the kernel of $\mathcal{A} - i\lambda^{\tau,k}$ is nontrivial. It will be more involved when the eigenvalues have multiplicity greater than 1. It can be characterized that the kernel and the orthogonal of $\mathcal{A} - i\lambda^{\tau,k}$ with respect to the inner product (5.2) are

$$\text{Ker}(\mathcal{A} - i\lambda^{\tau,k}) = \text{Span}\{U^{\tau,l} : \text{for all } l \in \mathbb{N} \text{ such that } \lambda^l = \lambda^k\},$$

and

$$\begin{aligned} \text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp = & \text{Span}\{U^{\delta,l} : \text{for all } \delta = \pm \text{ and } l \in \mathbb{N} \text{ such that } \lambda^l \neq \lambda^k\} \\ & \oplus \text{Span}\{U^{-\tau,l} : \lambda^l = \lambda^k\} \oplus \text{Null}(\mathcal{A}). \end{aligned}$$

Next, we define a bounded pseudo inverse of $\mathcal{A} - i\lambda^{\tau,k}$

$$(\mathcal{A} - i\lambda^{\tau,k})^{-1} : \text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp \longrightarrow \text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp,$$

by

$$(\mathcal{A} - i\lambda^{\tau,k})^{-1}U^{\delta,l} = \frac{1}{i\lambda^{\delta,l} - i\lambda^{\tau,k}}U^{\delta,l}, \quad \text{for any } U^{\delta,l} \text{ with } \lambda^l \neq \lambda^k, \quad (5.17)$$

$$(\mathcal{A} - i\lambda^{\tau,k})^{-1}U^{-\tau,l} = \frac{1}{-2i\lambda^{\tau,k}}U^{-\tau,l}, \quad \text{for any } U^{-\tau,l} \text{ with } \lambda^l = \lambda^k, \quad (5.18)$$

and

$$(\mathcal{A} - i\lambda^{\tau,k})^{-1}(\rho, v, -\rho)^T = \frac{1}{i\lambda^{\tau,k}}(\rho, v, -\rho)^T, \quad (5.19)$$

for any $(\rho, v, -\rho)^T \in \text{Null}(\mathcal{A})$ and $\tau, \delta \in \{+, -\}$. It is obvious that this pseudo inverse operator is a bounded operator. Consequently, the solutions to the system (5.16) are stated in the following lemma.

Lemma 5.1. *For each fixed acoustic modes $k \geq 1$ and $\tau \in \{+, -\}$, the solvability conditions of the system (5.16) are:*

(i) *If λ^k is a simple eigenvalue of (5.6), then the only solvability condition is that $i\mu^{\tau,k}$ must satisfy*

$$i\mu^{\tau,k} = \int_{\partial\Omega} g^{\tau,k} \Psi^k d\sigma_x - \langle F^{\tau,k} | U^{\tau,k} \rangle. \quad (5.20)$$

Under this condition, the solutions to (5.16) $V^{\tau,k}$ can be solved uniquely as

$$V^{\tau,k} = V_1^{\tau,k}, \quad (5.21)$$

where $V_1^{\tau,k} \in \text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp$.

(ii) *If λ^k is not a simple eigenvalue, then besides (5.20), further compatibility condition is needed: $F^{\tau,k}$ must satisfy:*

$$\int_{\partial\Omega} g^{\tau,k} \Psi^l d\sigma_x = \langle F^{\tau,k} | U^{\tau,l} \rangle, \quad \text{for } \lambda^l = \lambda^k \text{ with } k \neq l. \quad (5.22)$$

For this case, under these two conditions (5.20)-(5.22), the solutions to (5.16) $V^{\tau,k}$ can be determined modulo $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})$. In other words, $V^{\tau,k}$ can be uniquely represented as

$$V^{\tau,k} = \sum_{\lambda^k = \lambda^l} \langle V^{\tau,k} | U^{\tau,l} \rangle U^{\tau,l} + V_1^{\tau,k}, \quad (5.23)$$

where $V_1^{\tau,k} \in \text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp$.

Proof. For any $g^{\tau,k} \in H^{\frac{1}{2}}(\partial\Omega)$, there exists $\tilde{v}^{\tau,k} \in H^1(\Omega; \mathbb{R}^D)$, such that $\gamma \tilde{v}^{\tau,k} \cdot n = g^{\tau,k}$, where γ is the usual trace operator from $H^1(\Omega; \mathbb{R}^D)$ to $H^{\frac{1}{2}}(\partial\Omega)$. We define

$$\tilde{V}^{\tau,k} = V^{\tau,k} - (0, \tilde{v}^{\tau,k}, 0)^T. \quad (5.24)$$

Then $\tilde{V}^{\tau,k}$ has zero the normal velocity on the boundary $\partial\Omega$, thus is in the domain of \mathcal{A} . From (5.16), $\tilde{V}^{\tau,k}$ satisfies

$$(\mathcal{A} - i\lambda^{\tau,k})\tilde{V}^{\tau,k} = -(\mathcal{A} - i\lambda^{\tau,k})(0, \tilde{v}^{\tau,k}, 0) + i\mu^{\tau,k}U^{\tau,k} + F^{\tau,k}. \quad (5.25)$$

The solvability of (5.25) is that the righthand side must be in $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp$. Thus, the inner product of (5.25) with $U^{\tau,k}$ is zero, which gives (5.20), while the inner product with $U^{\tau,l}$ with $\lambda^k = \lambda^l, k \neq l$ gives (5.22). Under these conditions, by applying the psedu inverse operator

$(\mathcal{A} - i\lambda^{\tau,k})^{-1}$ defined in (5.17)-(5.19), we can uniquely solve $\tilde{V}^{\tau,k}$ in $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp$, denoted by $\tilde{V}_1^{\tau,k}$. However, the projection of $\tilde{V}^{\tau,k}$ on $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})$ is *not* determined. In other words,

$$\tilde{V}^{\tau,k} = \tilde{V}_1^{\tau,k} + \sum_{\lambda^k = \lambda^l} \langle \tilde{V}^{\tau,k} | U^{\tau,l} \rangle U^{\tau,l}.$$

Using (5.24), we get (5.23), where

$$V_1^{\tau,k} = \tilde{V}_1^{\tau,k} + (0, \tilde{v}^{\tau,k}, 0)^T - \sum_{\lambda^k = \lambda^l} \langle (0, \tilde{v}^{\tau,k}, 0)^T | U^{\tau,l} \rangle U^{\tau,l}.$$

In (5.23), the projection of $V^{\tau,k}$ on $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})$, i.e. the first term in the righthand side of (5.23), can *not* be determined. It is easy to see that the projection of $V^{\tau,k}$ on $\text{Ker}(\mathcal{A} - i\lambda^{\tau,k})^\perp$, i.e. $V_1^{\tau,k}$, is uniquely determined, although the lifting of the trace $g^{\tau,k}$ is not unique. \square

6. ANALYSIS OF THE KINETIC BOUNDARY LAYER EQUATION

In this section, we collect three results in kinetic equations which will be frequently used in this paper. The first two results are standard in kinetic theory:

Lemma 6.1. *The solvability condition for the linear kinetic equation $\mathcal{L}g = f$ is*

$$\langle f, \zeta(v) \rangle = 0, \quad \text{for } \zeta \in \text{Span}\{1, v, |v|^2\}. \quad (6.1)$$

The second result we will use is quoted from Lemma 4.4 in [3].

Lemma 6.2. *The components of $\langle \mathbf{A} \otimes \hat{\mathbf{A}} \rangle$ and $\langle \mathbf{B} \cdot \hat{\mathbf{B}} \rangle$ satisfy the following identities:*

$$\begin{aligned} \langle \mathbf{A}_{ij} \otimes \hat{\mathbf{A}}_{kl} \rangle &= \nu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{D}\delta_{ij}\delta_{kl}), \\ \langle \mathbf{B}_i \hat{\mathbf{B}}_j \rangle &= \frac{D+2}{2}\kappa\delta_{ij}, \end{aligned}$$

where ν and κ are given by (3.5).

The next result is about the linear kinetic boundary layer equation which will be used to determine the boundary conditions of the fluid variables. We define the kinetic boundary layer operator \mathcal{L}^{BL} , reflection boundary operator $L^{\mathcal{R}}$ and diffusive boundary operator $L^{\mathcal{D}}$ acting on functions $\{g^{\text{bb}}(x, v, \xi) : (x, v, \xi) \in \Omega^\delta \times \mathbb{R}^D \times \mathbb{R}_+\}$ as follows:

$$\mathcal{L}^{BL}g^{\text{bb}} := -(v \cdot \nabla_x d)\partial_\xi g^{\text{bb}} + \mathcal{L}g^{\text{bb}}, \quad (6.2)$$

where \mathcal{L} is the linearized Boltzmann operator defined in (2.8).

$$L^{\mathcal{R}}g^{\text{bb}} := \gamma_+ g^{\text{bb}} - L\gamma_- g^{\text{bb}}, \quad \text{and} \quad L^{\mathcal{D}}g^{\text{bb}} := \sqrt{2\pi}\chi \left[\langle \gamma_- g^{\text{bb}} \rangle_{\partial\Omega} - L\gamma_- g^{\text{bb}} \right].$$

Lemma 6.3. *Considering the following linear kinetic boundary layer equation of $g^{\text{bb}}(x, v, \xi)$ in half space:*

$$\begin{aligned} \mathcal{L}^{BL}g^{\text{bb}} &= S^{\text{bb}}, \quad \text{in } \xi > 0, \\ g^{\text{bb}} &\longrightarrow 0, \quad \text{as } \xi \rightarrow \infty, \end{aligned} \quad (6.3)$$

with boundary condition

$$L^{\mathcal{R}}g^{\text{bb}} = H^{\text{bb}}, \quad \text{on } \xi = 0, \quad v \cdot \mathbf{n} > 0. \quad (6.4)$$

In the above equations, the boundary source term H^{bb} is taken of the following form:

$$H^{\text{bb}} = -L^{\mathcal{R}}g + L^{\mathcal{D}}f, \quad (6.5)$$

where g and f are of the forms:

$$\begin{aligned} g = & \rho_g + \mathbf{u}_g \cdot \mathbf{v} + \theta_g \left(\frac{|\mathbf{v}|^2}{2} - \frac{D}{2} \right) \\ & - (\partial_\zeta \mathbf{u}^b \otimes \mathbf{n} : \hat{\mathbf{A}} + \partial_\zeta \theta^b \mathbf{n} \cdot \hat{\mathbf{B}}) + (\partial_{\pi^\alpha} \tilde{\mathbf{u}}^b \otimes \nabla_x \pi^\alpha : \hat{\mathbf{A}} + \partial_{\pi^\alpha} \tilde{\theta}^b \nabla_x \pi^\alpha \cdot \hat{\mathbf{B}}) \\ & + (\nabla_x \mathbf{u}^{\text{int}} : \hat{\mathbf{A}} + \nabla_x \theta^{\text{int}} \cdot \hat{\mathbf{B}}) + S_g, \end{aligned} \quad (6.6)$$

and

$$f = \rho_f + \mathbf{u}_f \cdot \mathbf{v} + \theta_f \left(\frac{|\mathbf{v}|^2}{2} - \frac{D}{2} \right) + S_f, \quad (6.7)$$

and where $S_g, S_f \in \text{Null}(\mathcal{L})^\perp$ are source terms.

Then there exists a solution $g^{\text{bb}}(x, v, \xi)$ of the equation (6.3) if and only if the following boundary conditions are satisfied by the fluid variables:

(i) On the boundary $\partial\Omega$, the normal components of velocity is

$$\mathbf{u}_g \cdot \mathbf{n} = \int_0^\infty \langle S^{\text{bb}} \rangle d\xi. \quad (6.8)$$

(ii) On the boundary $\partial\Omega$, the tangential components of velocities and temperature satisfy

$$\begin{aligned} [\mathbf{u}_f]^\text{tan} = & \frac{\nu}{\chi} \left[\partial_\zeta \mathbf{u}^b \right]^\text{tan} - \frac{\nu}{\chi} [2d(\mathbf{u}^{\text{int}}) \cdot \mathbf{n}]^\text{tan} - \frac{\nu}{\chi} \nabla_\pi [\tilde{\mathbf{u}}^b \cdot \mathbf{n}] \\ & + \left[\int_{v \cdot \mathbf{n} > 0} (L^\mathcal{D} S_f) v (v \cdot \mathbf{n}) M dv \right]^\text{tan} - \frac{1}{\chi} \langle (v \cdot \mathbf{n}) v S_g \rangle^\text{tan} + \frac{1}{\chi} \int_0^\infty \langle S^{\text{bb}} v \rangle^\text{tan} d\xi, \end{aligned} \quad (6.9)$$

and

$$\begin{aligned} \theta_f = & \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\zeta \theta^b - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_n \theta^{\text{int}} + \frac{\sqrt{2\pi}}{2(D+1)} \mathbf{u}_f \cdot \mathbf{n} + \frac{\sqrt{2\pi}}{D+1} \int_{v \cdot \mathbf{n} > 0} (L^\mathcal{D} S_f) |v|^2 (v \cdot \mathbf{n}) M dv \\ & - \frac{1}{(D+1)\chi} \langle (v \cdot \mathbf{n}) |v|^2 S_g \rangle + \frac{D+2}{D+1} \frac{1}{\chi} \int_0^\infty \langle S^{\text{bb}} (\frac{|v|^2}{D+2} - 1) \rangle d\xi, \end{aligned} \quad (6.10)$$

where kinematic viscosity ν and thermal conductivity κ are given by (3.5), \mathbf{u}^tan denotes the tangential components of the vector \mathbf{u} , and ∇_π denotes the tangential derivative.

Proof. The solvability conditions of the linear boundary layer equation (6.3) with boundary condition (6.4) are given by

$$\int_{v \cdot \mathbf{n} > 0} H^{\text{bb}} \eta(v) (v \cdot \mathbf{n}) M dv = - \int_0^\infty \langle S^{\text{bb}} \eta \rangle d\xi, \quad (6.11)$$

for all $\eta(v) \in \text{Null}(\mathcal{L})$ satisfying the condition:

$$\eta(R_x v) = \eta(v). \quad (6.12)$$

It is obvious that 1 and $|v|^2$ satisfy (6.12). If $\eta(v) = \sum_1^D a_i v_i$ satisfies (6.12), then necessarily

$$(v \cdot \mathbf{n}) \sum_{i=1}^D a_i \mathbf{n}_i = 0, \quad (6.13)$$

which implies that the vector $\mathbf{a} = (a_1, \dots, a_D)^\top$ is perpendicular to the outer normal vector \mathbf{n} .

The formula (6.8) can be derived by taking $\eta = 1$ in (6.11). Simple calculations show that

$$\begin{aligned} \int_{v \cdot \mathbf{n} > 0} H^{\text{bb}}(v) (v \cdot \mathbf{n}) M dv &= - \langle v \gamma g \rangle \cdot \mathbf{n} \\ &= - \mathbf{u}_g \cdot \mathbf{n} - \langle v S_g \rangle \cdot \mathbf{n}. \end{aligned}$$

Note that $S_g \in \text{Null}(\mathcal{L})^\perp$, hence (6.8) follows.

To prove (6.9), by taking $\eta = \sum_1^D a_i v_i$ in (6.11), we have

$$\int_{v \cdot n > 0} H^{bb}(a_i v_i)(n_j v_j) M dv = - \int_0^\infty \langle (a_i v_i) S^{bb} \rangle d\xi.$$

In other words,

$$\begin{aligned} & - \int_{v \cdot n > 0} (L^{\mathcal{R}} g)(a_i v_i)(n_j v_j) M dv + \int_{v \cdot n > 0} (L^{\mathcal{D}} f)(a_i v_i)(n_j v_j) M dv \\ & = - \int_0^\infty \langle S^{bb}(a_i v_i) \rangle d\xi. \end{aligned}$$

Simple calculations yield that

$$\int_{v \cdot n > 0} (L^{\mathcal{R}} g)(a_i v_i)(n_j v_j) M dv = \langle v_i v_j \gamma g \rangle a_i n_j = \langle A_{ij} \gamma g \rangle a_i n_j.$$

Using the definition of the viscosity ν in (3.5) and Lemma 6.2, we have

$$\begin{aligned} & \int_{v \cdot n > 0} (L^{\mathcal{R}} g)(a_i v_i)(n_j v_j) M dv \\ & = -\nu [\partial_\zeta u^b] \cdot a + \nu \left[(\nabla_x u^{\text{int}} + (\nabla_x u^{\text{int}})^\top) \cdot n \right] \cdot a + \nu \nabla_\pi [\tilde{u} \cdot n] \cdot a + \langle S_g A n \rangle \cdot a. \end{aligned}$$

Next, it can be calculated that

$$L^{\mathcal{D}} f = \left(\frac{D+1}{2} \theta_f - \frac{\sqrt{2\pi}}{2} u_f \cdot n \right) - u_f \cdot v + 2(u_f \cdot n) n \cdot v - \frac{1}{2} \theta_f |v|^2 + L^{\mathcal{D}} S_f, \quad (6.14)$$

where $R_x u_f = u_f - 2(u_f \cdot n)n$ is the reflection of u_f with respect to the normal $n(x)$. Thus

$$\int_{v \cdot n > 0} (L^{\mathcal{D}} f)(a_i v_i)(v \cdot n) M dv = -\frac{1}{\sqrt{2\pi}} (R_x u_f) \cdot a + \left[\int_{v \cdot n > 0} (L^{\mathcal{D}} S_f) A \cdot n M dv \right] \cdot a.$$

To prove (6.10), taking $\eta = |v|^2$ in (6.11), we have

$$\int_{v \cdot n > 0} H^{bb} |v|^2 (v \cdot n) M dv = - \int_0^\infty \langle |v|^2 S^{bb} \rangle d\xi.$$

It can be calculated that

$$\int_{v \cdot n > 0} (L^{\mathcal{R}} g)(v \cdot n) |v|^2 M dv = 2 \langle \gamma g B \rangle \cdot n + (D+2) \langle (v \cdot n) \gamma g \rangle.$$

Using the definition of the thermal conductivity κ in (3.5) and Lemma 6.2, we have

$$\begin{aligned} & \int_{v \cdot n > 0} L^{\mathcal{R}} g(v \cdot n) |v|^2 M dv \\ & = - (D+2) \kappa \partial_\zeta \theta^b + (D+2) \kappa \nabla_x \theta^{\text{int}} \cdot n + (D+2) u_g \cdot n + \langle (v \cdot n) |v|^2 S_g \rangle. \end{aligned}$$

Furthermore,

$$\begin{aligned} & \int_{v \cdot n > 0} (L^{\mathcal{D}} f)(v \cdot n) |v|^2 M dv \\ & = \frac{1}{2} u_f \cdot n - \frac{D+1}{\sqrt{2\pi}} \theta_f + \int_{v \cdot n > 0} (L^{\mathcal{D}} S_f)(v \cdot n) |v|^2 M dv. \end{aligned}$$

As showed in (6.13), the vector a is perpendicular to n . Thus the resulting vector of inner product with a is the tangential part. Finally, noticing $[R_x u_f]^{\text{tan}} = [u_f]^{\text{tan}}$, we then finish the proof of the Lemma 6.3. \square

7. APPROXIMATE EIGENFUNCTIONS-EIGENVALUES

7.1. Motivation. We define the operators \mathcal{L}_ε and $\mathcal{L}_\varepsilon^*$ as

$$\mathcal{L}_\varepsilon := \frac{1}{\varepsilon} \mathcal{L} - v \cdot \nabla_x, \quad \mathcal{L}_\varepsilon^* := \frac{1}{\varepsilon} \mathcal{L} + v \cdot \nabla_x.$$

Formally, \mathcal{L}_ε and $\mathcal{L}_\varepsilon^*$ are “dual” in the following sense:

$$\langle \mathcal{L}_\varepsilon^* g^*, g \rangle = \langle g^*, \mathcal{L}_\varepsilon g \rangle, \quad (7.1)$$

provided that g^* satisfies the Maxwell reflection boundary condition

$$\gamma_- g^* = (1 - \alpha) L \gamma_+ g^* + \alpha \langle \gamma_+ g^* \rangle_{\partial\Omega} \quad \text{on } \Sigma_-, \quad (7.2)$$

and g satisfies the dual boundary condition

$$\gamma_+ g = (1 - \alpha) L \gamma_- g + \alpha \langle \gamma_- g \rangle_{\partial\Omega} \quad \text{on } \Sigma_+. \quad (7.3)$$

If g_ε is the fluctuation defined in (2.18), then g_ε obeys the scaled Boltzmann equation (2.20) in which $\mathcal{L}_\varepsilon^* g_\varepsilon$ appears and g_ε satisfies the boundary condition (7.2). Then from (7.1), $\mathcal{L}_\varepsilon g_\varepsilon^{BL}$ appears in the weak formulation of the Boltzmann equation if we take g_ε^{BL} as a test function. Thus, it is natural to construct eigenfunctions and eigenvalues of \mathcal{L}_ε satisfying the dual boundary condition (7.3). Specifically, we consider the kinetic eigenvalue problem:

$$\mathcal{L}_\varepsilon g_\varepsilon^{BL} = -i \lambda_\varepsilon^{BL} g_\varepsilon^{BL}, \quad (7.4)$$

with g_ε^{BL} satisfying the dual Maxwell boundary condition (7.3), where the accommodation coefficient α takes the value $\alpha_\varepsilon = \sqrt{2\pi} \chi \sqrt{\varepsilon}$. By doing so, formally the equation (2.20) becomes an ordinary differential equation of $b_\varepsilon = \int_\Omega \langle g_\varepsilon, g_\varepsilon^{BL} \rangle dx$:

$$\frac{d}{dt} b_\varepsilon + \frac{i \lambda_\varepsilon^{BL}}{\varepsilon} b_\varepsilon = c_\varepsilon.$$

To solve the eigenvalue problem (7.4) and (7.3), a key observation is that the solutions must include interior and two boundary layer terms: the *fluid viscous boundary layer* with thickness $\sqrt{\varepsilon}$, and the *kinetic Knudsen layer* with thickness ε . We make the ansatz of g_ε^{BL} and λ_ε^{BL} as

$$g_\varepsilon^{BL} = \sum_{m \geq 0} \left[g_m^{\text{int}}(x, v) + g_m^{\text{b}}(\pi(x), \frac{d(x)}{\sqrt{\varepsilon}}, v) \right] \varepsilon^{\frac{m}{2}} + \sum_{m \geq 1} g_m^{\text{bb}}(\pi(x), \frac{d(x)}{\varepsilon}, v) \varepsilon^{\frac{m}{2}}, \quad (7.5)$$

and

$$\lambda_\varepsilon^{BL} = \sum_{m \geq 0} \lambda_m \varepsilon^{\frac{m}{2}}. \quad (7.6)$$

Each g_m^{b} and g_m^{bb} are defined in Ω^δ , the δ -tubular neighborhood of $\partial\Omega$ in Ω , where $\delta > 0$ is the small number defined in Lemma 4.1, the projection π is defined in (4.1). After rescaling by $\sqrt{\varepsilon}$ and ε respectively,

$$g_m^{\text{b}}, g_m^{\text{bb}} : (\partial\Omega \times \mathbb{R}^+) \times \mathbb{R}^D \longrightarrow \mathbb{R}.$$

Both g_m^{b} and g_m^{bb} will vanish in the outside of Ω^δ . Thus g_m^{b} and g_m^{bb} are required to be rapidly decreasing to 0 in the ζ and ξ respectively, which are defined by $\zeta = \frac{d(x)}{\sqrt{\varepsilon}}$ and $\xi = \frac{d(x)}{\varepsilon}$.

In the ansatz (7.5), g_ε^{BL} consists three types of terms: the interior terms g_m^{int} , the fluid viscous boundary layer terms g_m^{b} , and the kinetic Knudsen layer terms g_m^{bb} . They are coupled through the boundary condition (7.3).

7.2. Statement of the Proposition. Now we state the proposition which can be considered as a kinetic analogue of the Proposition 2 in [10].

Proposition 7.1. *Let Ω be a C^2 bounded domain of \mathbb{R}^D and the accommodation coefficient $\alpha_\varepsilon = \sqrt{2\pi}\chi\sqrt{\varepsilon}$. Then, for every acoustic mode $k \geq 1$, non-negative integer N , and each $\tau \in \{+, -\}$, there exists approximate eigenfunctions $g_{\varepsilon,N}^{\tau,k}$ and eigenvalues $-i\lambda_{\varepsilon,N}^{\tau,k}$ of \mathcal{L}_ε , and error terms $R_{\varepsilon,N}^{\tau,k}$ and $r_{\varepsilon,N}^{\tau,k}$ respectively, such that*

$$\mathcal{L}_\varepsilon g_{\varepsilon,N}^{\tau,k} = -i\lambda_{\varepsilon,N}^{\tau,k} g_{\varepsilon,N}^{\tau,k} + R_{\varepsilon,N}^{\tau,k}, \quad (7.7)$$

and $g_{\varepsilon,N}^{\tau,k}$ satisfy the approximate dual Maxwell boundary condition:

$$L^{\mathcal{R}} g_{\varepsilon,N}^{\tau,k} = \sqrt{\varepsilon} L^{\mathcal{D}} g_{\varepsilon,N}^{\tau,k} + r_{\varepsilon,N}^{\tau,k} \quad \text{on } \Sigma_+. \quad (7.8)$$

Moreover, there exists complex numbers $\lambda_1^{\tau,k}$, such that $i\lambda_{\varepsilon,N}^{\tau,k}$ has the following expansions:

$$i\lambda_{\varepsilon,N}^{\tau,k} = i\lambda_0^{\tau,k} + i\lambda_1^{\tau,k} \sqrt{\varepsilon} + O(\varepsilon), \quad \text{with } \operatorname{Re}(i\lambda_1^{\tau,k}) < 0. \quad (7.9)$$

Furthermore, for all $1 < r, p \leq \infty$, we have error estimates:

$$\|R_{\varepsilon,N}^{\tau,k}\|_{L^r(dx, L^p(a^{1-p} M dv))} = O(\sqrt{\varepsilon}^{N-1}), \quad (7.10)$$

and

$$\|g_{\varepsilon,N}^{\tau,k} - g_0^{\tau,k,\text{int}}\|_{L^r(dx, L^p(a^{1-p} M dv))} = O(\varepsilon^{\frac{1}{2r}}). \quad (7.11)$$

where $g_0^{\tau,k,\text{int}}$ is defined in (5.9). We also have the boundary error estimates:

$$\|r_{\varepsilon,N}^{\tau,k}\|_{L^r(d\sigma_x, L^p(a^{1-p} M dv))} = O(\sqrt{\varepsilon}^{N+1}). \quad (7.12)$$

7.3. Main Idea of the Proof. For each non-negative integer m , g_m^{int} is decomposed as hydrodynamic part $\mathcal{P} g_m^{\text{int}}$, i.e. the projection on $\text{Null}(\mathcal{L})$, and kinetic part $\mathcal{P}^\perp g_m^{\text{int}}$, the projection on $\text{Null}(\mathcal{L})^\perp$. The hydrodynamic part is given by

$$\mathcal{P} g_m^{\text{int}} = \rho_m^{\text{int}} + v \cdot u_m^{\text{int}} + \left(\frac{|v|^2}{2} - \frac{D}{2}\right) \theta_m^{\text{int}} = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U_m^{\text{int}},$$

where $U_m^{\text{int}} = (\rho_m^{\text{int}}, u_m^{\text{int}}, \theta_m^{\text{int}})^\top$ is called the fluid variables of g_m^{int} . It can be shown that the coefficients of $\mathcal{P}^\perp g_m^{\text{int}}$ are in terms of $U_{m'}^{\text{int}}$ and their derivatives for $m' < m$. Thus, we need only to solve U_n^{int} for all integers $n \leq m$ to determine g_m^{int} . Similar notations will be used for g_m^{b} , and for the same reason we also need only to solve U_m^{b} .

We put the ansatz into the equation (7.4), then collect the same order terms. The leading order term g_0^{int} is hydrodynamic, which means g_0^{int} is completely determined by U_0^{int} . we can derive that U_0^{int} satisfies the equation

$$\mathcal{A} U_0^{\text{int}} = i\lambda_0 U_0^{\text{int}}. \quad (7.13)$$

For (7.13) there are two cases:

- **Case 1:** $\lambda_0 \neq 0$, by comparing the equations (7.13) and (5.8), $i\lambda_0$ is an eigenvalue of the acoustic operator \mathcal{A} , i.e. $\lambda_0 = \lambda^{\tau,k}$ and $U_0^{\text{int}} = U^{\tau,k}$, where $k \geq 0$ is the acoustic modes, and τ denotes either $+$ or $-$. Starting from here, we can construct the boundary layer g_ε^{BL} which we call the boundary layer in the **acoustic modes**.
- **Case 2:** $\lambda_0 = 0$, which implies that $U_0^{\text{int}} \in \text{Ker}(\mathcal{A})$, i.e. $\rho_0^{\text{int}} + \theta_0^{\text{int}} = 0$ and $\nabla_x \cdot u_0^{\text{int}} = 0$. Starting from here, we can construct the boundary layer g_ε^{BL} which we call the boundary layer in the **incompressible modes**.

Because the main goal of the current paper is about how the acoustic waves and the boundary layers interact in the incompressible Navier-Stokes limit of the Boltzmann equation, we only consider the **Case 1**, the kinetic-fluid boundary layers in the acoustic modes for each $k \geq 0$ and each τ . Consequently, we add superscript τ, k for each term in the ansatz. In the forthcoming paper [25], we will investigate the higher order acoustic limit of the Boltzmann equation, where we need to analyze the boundary layers in incompressible modes, i.e. **Case 2**.

The basic strategy to solve all terms in the ansatz is the following: (for the simplicity of notation, we don't write the upper index τ)

- 1**, $g_m^{k,bb}$ satisfies the linear kinetic boundary layer equation (6.3). Applying Lemma 6.3, the solvability conditions for $g_m^{k,bb}$ give the normal boundary condition $[u_m^{k,int} + u_m^{k,b}] \cdot n$ and the tangential boundary condition $[u_{m-1}^{k,b} - \frac{\nu}{\chi} \partial_\zeta u_{m-1}^{k,b}]^{\tan} + [u_{m-1}^{k,int}]^{\tan}$ and $\theta_{m-1}^{k,b} - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\zeta \theta_{m-1}^{k,b} + \theta_{m-1}^{k,int}$;
- 2**, $U_m^{k,b}$ satisfies the ODE system like (8.96), where the normal boundary acoustic operator \mathcal{A}^d is defined in (8.1). Solving $U_m^{k,b}$ includes two steps: First, projecting the ODE system of $U_m^{k,b}$ on $\text{Null}^\perp(\mathcal{A}^d)$ to get the first order ODE satisfied by $\rho_m^{k,b} + \theta_m^{k,b}$ and $u_m^{k,b} \cdot \nabla_x d$ (thus only need one boundary condition at $\zeta = \infty$). The next, projecting on $\text{Null}(\mathcal{A}^d)$, to solve $u_{m-1}^{k,b} \cdot \nabla_x \pi$ and $\theta_{m-1}^{k,b}$ which satisfy second ODE. Thus besides the condition at $\zeta = \infty$, another boundary condition at $\zeta = 0$ is needed. This is given by the solvability condition of $g_m^{k,bb}$ in **1**.
- 3**, $U_m^{k,int}$ and $i\lambda_m^k$ can be solved by applying Lemma 5.1, where only the boundary condition in the normal direction $u_m^{k,int} \cdot n$ is needed which can be known from $u_m^{k,b} \cdot n$ since their summation $[u_m^{k,int} + u_m^{k,b}] \cdot n$ is found in (1), and $u_m^{k,b} \cdot n$ is already known in **2**;
- 4**, If the multiplicity of the eigenvalue λ_0^k is greater than 1, applying Lemma 5.1, $U_m^{k,int}$ can be only solved modulo $\text{Ker}(\mathcal{A} - i\lambda_0^k)$. The components of $U_m^{k,int}$ in $\text{Ker}(\mathcal{A} - i\lambda_0^k)$ will be solved by applying Lemma (5.1) again in later rounds.

8. PROOF OF PROPOSITION 7.1: CONSTRUCTION OF BOUNDARY LAYERS

In this section, we construct the kinetic-fluid boundary layers corresponding to the accommodation coefficient $\alpha_\epsilon = \sqrt{2\pi}\chi\sqrt{\epsilon}$. As mentioned in the previous section, we make the ansatz for g_ϵ^{BL} and λ_ϵ^{BL} as in (7.5) and (7.6). Then we formally plug the ansatz (7.5) into the equation (7.4) and then collect the terms with the same order of ϵ in the interior, the viscous boundary layer, and the Knudsen layer respectively. The following calculations will be frequently used:

$$\begin{aligned} v \cdot \nabla_x g^b(\pi(x), \frac{d(x)}{\sqrt{\epsilon}}) &= (v \cdot \nabla_x \pi^\alpha) \partial_{\pi^\alpha} g^b + \frac{1}{\sqrt{\epsilon}} (v \cdot \nabla_x d) \partial_\zeta g^b, \\ v \cdot \nabla_x g^{bb}(\pi(x), \frac{d(x)}{\epsilon}) &= (v \cdot \nabla_x \pi^\alpha) \partial_{\pi^\alpha} g^{bb} + \frac{1}{\epsilon} (v \cdot \nabla_x d) \partial_\xi g^{bb}. \end{aligned}$$

8.1. Normal Boundary Acoustic Operator. A key role played in the analysis of the viscous boundary layer is the so-called normal boundary acoustic operator \mathcal{A}^d of the viscous boundary fluid variables $U^b = (\rho^b, u^b, \theta^b)^\top$:

$$\mathcal{A}^d U^b := \begin{pmatrix} \partial_\zeta (u^b \cdot \nabla_x d) \\ \partial_\zeta (\rho^b + \theta^b) \nabla_x d \\ \frac{2}{D} \partial_\zeta (u^b \cdot \nabla_x d) \end{pmatrix}. \quad (8.1)$$

The null space of \mathcal{A}^b and its orthogonal space are

$$\begin{aligned} \text{Null}(\mathcal{A}^d) &= \{(\rho^b, u^b, \theta^b)^\top \in L^2(dx; \Omega^\delta) : \rho^b + \theta^b = 0, u^b \cdot \nabla_x d = 0\}, \\ \text{Null}^\perp(\mathcal{A}^d) &= \{(\rho^b, u^b, \theta^b)^\top \in L^2(dx; \Omega^\delta) : \theta^b = \frac{2}{D} \rho^b, u^b \cdot \nabla_x \pi = 0\}, \end{aligned}$$

where the orthogonality is with respect to the inner product endowed on $L^2(dx; \Omega_\delta)$:

$$\langle \tilde{U}^b, U^b \rangle_{L^2(\Omega_\delta)} := \int_{\Omega_\delta} (\tilde{\rho}^b \overline{\rho^b} + \tilde{u}^b \cdot \overline{u^b} + \frac{D}{2} \tilde{\theta}^b \overline{\theta^b}) dx.$$

The projections from $L^2(\Omega_\delta)$ to $\text{Null}(\mathcal{A}^d)$ and $\text{Null}(\mathcal{A}^d)^\perp$ are defined as

$$\begin{aligned} U^b &= \Pi^b U^b + (I - \Pi^b) U^b \\ &:= \begin{pmatrix} \frac{2}{D+2} \rho^b - \frac{D}{D+2} \theta^b \\ (\mathbf{u}^b \cdot \nabla_x \pi^\alpha) \nabla_x \pi^\alpha \\ \frac{D}{D+2} \theta^b - \frac{2}{D+2} \rho^b \end{pmatrix} + \begin{pmatrix} \frac{D}{D+2} (\rho^b + \theta^b) \\ (\mathbf{u}^b \cdot \nabla_x \mathbf{d}) \nabla_x \mathbf{d} \\ \frac{2}{D+2} (\rho^b + \theta^b) \end{pmatrix}. \end{aligned} \quad (8.2)$$

We remark that the normal boundary acoustic operator \mathcal{A}^d and its Null and Null orthogonal spaces appear in other places to play a key role. For example, the zero viscosity limit of compressible NSF equations with boundary, see [11].

8.2. Preparations. Before we start the induction, we solve g_0^{int} , g_0^b and $i\lambda_0$. First, the terms of order $O(\varepsilon^{-1})$ in the interior and viscous boundary layers in the equation (7.4) yield

$$\mathcal{L}g_0^{\text{int}} = 0 \quad \text{and} \quad \mathcal{L}g_0^b = 0,$$

which imply that g_0^{int} and g_0^b are hydrodynamic, i.e.

$$g_0^{\text{int}}(x, v) = \rho_0^{\text{int}} + v \cdot \mathbf{u}_0^{\text{int}} + \left(\frac{|v|^2}{2} - \frac{D}{2}\right) \theta_0^{\text{int}} = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U_0^{\text{int}},$$

and

$$g_0^b(\pi(x), \zeta, v) = \rho_0^b + v \cdot \mathbf{u}_0^b + \left(\frac{|v|^2}{2} - \frac{D}{2}\right) \theta_0^b = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U_0^b.$$

We denote above expressions as $g_0^{\text{int}} = I_0(U_0^{\text{int}})$ and $g_0^b = B_0(U_0^b)$. Here as operators, $I_0 = B_0$, we use different notations to emphasize that one is for interior variable, the other for viscous boundary variable. The fluid variables U_0^{int} and U_0^b are to be determined. To solve them we need to know the equations satisfied by them and their boundary conditions. It is easy to know from the order $O(\sqrt{\varepsilon}^{-1})$ of the interior part that g_1^{int} is also hydrodynamic, i.e. $g_1^{\text{int}} = I_0(U_1^{\text{int}})$.

8.3. Induction: Round 0. Now we start our induction arguments, each round includes considering the kinetic boundary layer, viscous boundary layer and interior terms alternately.

Step 1: Order $O(\sqrt{\varepsilon}^{-2})$ in the kinetic boundary layer.

The order $O(\sqrt{\varepsilon}^{-2})$ in the boundary condition (7.3) gives

$$L^{\mathcal{R}} \tilde{g}_0 = 0, \quad (8.3)$$

where we use the notation $\tilde{g} = g^{\text{int}} + g^b$. Because g_0^{int} and g_0^b are hydrodynamic, then

$$[\mathbf{u}_0^{\text{int}} + \mathbf{u}_0^b] \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega. \quad (8.4)$$

Step 2: Order $O(\sqrt{\varepsilon}^{-1})$ in the viscous boundary layer.

Next, the order $O(\sqrt{\varepsilon}^{-1})$ in the viscous boundary layer reads

$$\begin{aligned} \mathcal{L}g_1^b &= v \cdot \nabla_x \mathbf{d} \partial_\zeta g_0^b \\ &= \partial_\zeta \mathbf{u}_0^b \otimes \nabla_x \mathbf{d} : \mathbf{A} + \partial_\zeta \theta_0^b \nabla_x \mathbf{d} \cdot \mathbf{B} + \mathcal{A}^d U_0^b \cdot (1, v, \frac{|v|^2}{2} - \frac{D}{2}). \end{aligned} \quad (8.5)$$

Lemma 6.1 implies that the solvability condition for equation (8.5) is $\mathcal{A}^d U_0^b = 0$, i.e. U_0^b lies in the kernel of the normal boundary acoustic operator:

$$\partial_\zeta (\rho_0^b + \theta_0^b) \nabla_x \mathbf{d} = 0 \quad \text{and} \quad \partial_\zeta (\mathbf{u}_0^b \cdot \nabla_x \mathbf{d}) = 0.$$

from which we deduce that $\rho_0^b + \theta_0^b$ is constant in ζ . Since $\rho_0^b + \theta_0^b \rightarrow 0$ as $\zeta \rightarrow \infty$, then

$$\rho_0^b + \theta_0^b = 0. \quad (8.6)$$

Similarly, we have

$$\mathbf{u}_0^b \cdot \nabla_x \mathbf{d} = 0, \quad (8.7)$$

which also gives that on the boundary $\partial\Omega$,

$$\mathbf{u}_0^b(x, \zeta=0) \cdot \mathbf{n} = 0. \quad (8.8)$$

Combining (8.4) with the condition (8.8), we deduce that

$$\mathbf{u}_0^{\text{int}} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (8.9)$$

Under these conditions, g_1^b can be expressed as

$$\begin{aligned} g_1^b &= B_0(U_1^b) + B_1(U_0^b) \\ &:= (1, v, \frac{|v|^2}{2} - \frac{D}{2})U_1^b + \{\partial_\zeta \mathbf{u}_0^b \otimes \nabla_x \mathbf{d} : \widehat{\mathbf{A}} + \partial_\zeta \theta_0^b \nabla_x \mathbf{d} \cdot \widehat{\mathbf{B}}\}. \end{aligned} \quad (8.10)$$

Note that B_1 is a linear operator.

Step 3: Order $O(\sqrt{\varepsilon^0})$ in the interior.

To find the equation satisfied by U_0^{int} , we consider the order $O(\sqrt{\varepsilon^0})$ in the interior part:

$$\mathcal{L}g_2^{\text{int}} = v \cdot \nabla_x g_0^{\text{int}} - i\lambda_0 g_0^{\text{int}}. \quad (8.11)$$

Using the boundary condition (8.9) which means that U_0^{int} is in the domain of the acoustic operator \mathcal{A} , the solvability of (8.11) gives

$$\mathcal{A}U_0^{\text{int}} = i\lambda_0 U_0^{\text{int}}, \quad (8.12)$$

which is a first order linear hyperbolic system with the boundary condition (8.9). If $i\lambda_0 = 0$, (8.12) is the so-called acoustic system whose solutions U_0^{int} satisfies the incompressible and Boussinesq relations. This case will be treated in a separate paper [25].

If $i\lambda_0 \neq 0$, then from the discussions in section 5, especially (5.8), we know that the system (8.12) has a family of solutions: for each $k \in \mathbb{N}$, U_0^{int} are eigenvectors of \mathcal{A} , and can be constructed from the eigenvectors of the Laplace operator with Neumann boundary condition, i.e.

$$U_0^{\text{int}} = U_0^{\tau, k}, \quad \text{and} \quad \lambda_0 = \tau \lambda^k, \quad k = 1, 2, \dots \quad (8.13)$$

where τ denotes the signs either $+$ or $-$, see the discussion of the spectrum of \mathcal{A} in section 5, in particular (5.6), (5.7) and (5.8). Consequently, every term in the ansatz (7.5) and (7.6) depends on the choice of $k \in \mathbb{N}$ and τ .

Remark: (8.13) is the building-block of the construction of the approximate eigenvector $g_{\varepsilon, N}^k$ and eigenvalues $\lambda_{\varepsilon, N}^k$ for any $k, N \in \mathbb{N}$. It means that the leading order term of $(g_{\varepsilon, N}^k, \lambda_{\varepsilon, N}^k)$ is in acoustic modes. For this reason, we call $g_{\varepsilon, N}^k$ the boundary layer in *acoustic* regime.

Now we can represent g_2^{int} as

$$\begin{aligned} g_2^{\text{int}} &= I_0(U_2^{\tau, k, \text{int}}) + I_2(U_0^{\tau, k, \text{int}}) \\ &:= U_2^{\tau, k, \text{int}} \cdot (1, v, \frac{|v|^2}{2} - \frac{D}{2}) + \{\nabla_x \mathbf{u}_0^{\tau, k, \text{int}} : \widehat{\mathbf{A}}(v) + \nabla_x \theta_0^{\tau, k, \text{int}} \cdot \widehat{\mathbf{B}}(v)\}, \end{aligned} \quad (8.14)$$

where $\mathbf{u}_0^{\tau, k, \text{int}} = \sqrt{\frac{D+2}{2D}} \frac{\nabla_x \Psi^k}{i\lambda_0^{\tau, k}}$ and $\theta_0^{\tau, k, \text{int}} = \sqrt{\frac{2}{D(D+2)}} \Psi^k$, and Ψ^k is defined in (5.6).

Remark: For the notational simplicity, from now on we drop τ on the subscript, unless specifically mentioned.

8.4. Induction: Round 1. Now we move to round 1 which studies the kinetic boundary layer, viscous boundary layer and interior terms *alternately*.

Step 1: Order $O(\sqrt{\varepsilon^{-1}})$ in the kinetic boundary layer.

The order $O(\sqrt{\varepsilon}^{-1})$ of the kinetic boundary layer in the ansatz gives that $g_1^{k,bb}(x, v, \xi)$ obeys the following linear kinetic boundary layer equation in $\Omega^\delta \times \mathbb{R}^D \times \mathbb{R}_+$: (recalling the definition of \mathcal{L}^{BL} in (6.2))

$$\begin{aligned} \mathcal{L}^{BL} g_1^{k,bb} &= 0, \quad \text{in } \xi > 0, \\ g_1^{k,bb} &\longrightarrow 0, \quad \text{as } \xi \rightarrow \infty, \end{aligned} \quad (8.15)$$

with boundary condition at $\xi = 0$

$$L^{\mathcal{R}} g_1^{k,bb} = H_1^{k,bb}, \quad \text{on } \xi = 0, \quad v \cdot n(x) > 0, \quad (8.16)$$

where $H_1^{k,bb}$ is of the form: for $x \in \partial\Omega$, $v \cdot n(x) > 0$,

$$\begin{aligned} H_1^{k,bb}(x, v) &= -L^{\mathcal{R}} \tilde{g}_1^k + L^{\mathcal{D}} \tilde{g}_0^k \\ &:= \tilde{h}_0^{bb}(U_1^{\text{int}}, U_1^b) + \tilde{h}_1^{bb}(U_0^{\text{int}}, U_0^b). \end{aligned} \quad (8.17)$$

Here $\tilde{h}_0^{bb}(U_1^{\text{int}}, U_1^b)$ is a linear function of U_1^{int} and U_1^b , and $\tilde{h}_1^{bb}(U_0^{\text{int}}, U_0^b)$ is a linear function of U_0^{int} and U_0^b . More specifically,

$$\begin{aligned} \tilde{h}_0^{bb}(U_1^{\text{int}}, U_1^b) &:= -L^{\mathcal{R}}(I_0(U_1^{\text{int}}) + B_0(U_1^b)) \\ &= -2(v \cdot n)(\tilde{u}_1^k \cdot n), \end{aligned} \quad (8.18)$$

and

$$\begin{aligned} \tilde{h}_1^{bb}(U_0^{\text{int}}, U_0^b) &:= -L^{\mathcal{R}} B_1(U_0^b) + L^{\mathcal{D}}(I_0(U_0^{\text{int}}) + B_0(U_0^b)) \\ &= L^{\mathcal{R}}(\partial_\xi u_0^{k,b} \otimes n : \hat{A} + \partial_\xi \theta_0^{k,b} n \cdot \hat{B}) \\ &\quad - v \cdot [\tilde{u}_0^k \cdot \nabla_x \pi^\alpha] \nabla_x \pi^\alpha + \left(\frac{D+1}{2} - \frac{|v|^2}{2}\right) \tilde{\theta}_0^k + (n \cdot v - \frac{\sqrt{2\pi}}{2})(\tilde{u}_0^k \cdot n). \end{aligned} \quad (8.19)$$

The boundary conditions for the tangential components of $u_0^{k,b}$ and $\theta_0^{k,b}$ can be derived from the solvability condition of the above kinetic boundary layer equations. The formulas (6.9) and (6.10) of Lemma 6.3 give that on the boundary $\partial\Omega$,

$$[u_0^{k,b} - \frac{\nu}{\chi} \partial_\xi u_0^{k,b}]^{\text{tan}} = -[u_0^{k,\text{int}}]^{\text{tan}} \quad \text{and} \quad \theta_0^{k,b} - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\xi \theta_0^{k,b} = -\theta_0^{k,\text{int}}, \quad (8.20)$$

from which we can deduce the boundary conditions for $[u_0^{k,b}]^{\text{tan}}$ and $\theta_0^{k,b}$ because $u_0^{k,\text{int}}$ and $\theta_0^{k,\text{int}}$ are already determined, thus their boundary values are known. Furthermore, from (6.8) of Lemma 6.3, we also have

$$[u_1^{k,\text{int}} + u_1^{k,b}] \cdot n = 0, \quad \text{on } \partial\Omega, \quad (8.21)$$

which implies $\tilde{h}_0^{bb}(U_1^{\text{int}}, U_1^b) = 0$. Furthermore, from the boundary condition (8.20)

$$\begin{aligned} \tilde{h}_1^{bb}(U_0^{\text{int}}, U_0^b) &= L^{\mathcal{R}}(\partial_\xi u_0^{k,b} \otimes n : \hat{A} + \partial_\xi \theta_0^{k,b} n \cdot \hat{B}) \\ &\quad - \frac{\nu}{\chi} v \cdot [\partial_\xi u_0^{k,b}]^{\text{tan}} + \left(\frac{D+1}{2} - \frac{|v|^2}{2}\right) \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\xi \theta_0^{k,b}, \end{aligned} \quad (8.22)$$

which implies that although *formally*, $g_1^{k,bb}$ depends on $g_1^{k,\text{int}}$ and $g_1^{k,b}$ which have not been fully determined at this stage, it in fact depends only on the boundary values of $U_0^{k,b}$, thus once we solve $U_0^{k,b}$, we can solve $g_1^{k,bb}$ *completely*. This will be finished at the end of the **Step 2**, see (8.37).

Step 2: Order $O(\sqrt{\varepsilon}^0)$ in the viscous boundary layer.

The equations satisfied by $[u_0^{k,b}]^{\text{tan}}$ and $\theta_0^{k,b}$ can be derived by considering the order $O(\sqrt{\varepsilon}^0)$ of the viscous boundary layer:

$$\mathcal{L} g_2^{k,b} = v \cdot \nabla_x d \partial_\xi g_1^{k,b} + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} g_0^{k,b} - i \lambda_0^k g_0^{k,b}. \quad (8.23)$$

Lemma 6.1 implies that the projection of the righthand side of (8.23) must be in the null space of \mathcal{L} . We first use the expression (8.10) of $g_1^{k,b}$ to calculate the projection of $v \cdot \nabla_x d \partial_\zeta g_1^{k,b}$ onto $\text{Null}(\mathcal{L})$. It is easy to see that components of $\mathcal{P}(\partial_i d \partial_m d \partial_\zeta^2 (u_0^{k,b})_l v_i \hat{A}_{ml})$ on 1 and $\frac{|v|^2}{2} - \frac{D}{2}$ are zeros, here \mathcal{P} is defined in (2.11). Applying Lemma 6.2, we get

$$\mathcal{P} \left(\partial_i d \partial_m d \partial_\zeta^2 (u_0^{k,b})_l v_i \hat{A}_{ml} \right) = \left[\nu \partial_\zeta^2 u_0^{k,b} + \nu \left(1 - \frac{2}{D}\right) \partial_\zeta^2 (u_0^{k,b} \cdot \nabla_x d) \nabla_x d \right] \cdot v.$$

Similarly, the components of $\mathcal{P}(\partial_i d \partial_j d \partial_\zeta^2 \theta_0^{k,b} v_i \hat{B}_j)$ on 1 and v are zeros. Then applying Lemma 6.2 again, we have

$$\mathcal{P} \left(\partial_i d \partial_j d \partial_\zeta^2 \theta_0^{k,b} v_i \hat{B}_j \right) = \frac{D+2}{D} \kappa \partial_\zeta^2 \theta_0^{k,b} \left(\frac{|v|^2}{2} - \frac{D}{2} \right),$$

where kinematic viscosity ν and thermal conductivity κ are given by (3.5). Based on above calculations the solvability conditions for (8.23) are a system of second order ordinary differential equations in ζ :

$$-\mathcal{A}^d U_1^{k,b} = (\mathcal{A}^\pi + \mathcal{D}^d - i\lambda_0^k) U_0^{k,b}, \quad (8.24)$$

where the tangential acoustic operator \mathcal{A}^π and the normal diffusive operator \mathcal{D}^d are defined as

$$\mathcal{A}^\pi U^b := \begin{pmatrix} \text{div}_\pi(u^b \cdot \nabla_x \pi) \\ \partial_{\pi^\alpha}(\rho^b + \theta^b) \nabla_x \pi^\alpha \\ \frac{2}{D} \text{div}_\pi(u^b \cdot \nabla_x \pi) \end{pmatrix}, \quad \mathcal{D}^d U^b := \begin{pmatrix} 0 \\ \nu \partial_\zeta^2 u^b + \nu \left(1 - \frac{2}{D}\right) \partial_\zeta^2 (u^b \cdot \nabla_x d) \nabla_x d \\ \frac{D+2}{D} \kappa \partial_\zeta^2 \theta^b \end{pmatrix}, \quad (8.25)$$

for $U^b = (\rho^b, u^b, \theta^b)^\top$. Here we use the notation $\text{div}_\pi(u^b \cdot \nabla_x \pi) = \partial_{\pi^\alpha}(u^b \cdot \nabla_x \pi^\alpha)$. Recall the normal acoustic operator \mathcal{A}^d is defined in (8.1).

The ODE system (8.24) can be solved as follows: projecting the system (8.24) on $\text{Null}(\mathcal{A}^d)$ and $\text{Null}(\mathcal{A}^d)^\perp$ respectively, the projection on $\text{Null}(\mathcal{A}^d)$ gives the first order equations of $\rho_1^b + \theta_1^b$ and $u_1^b \cdot \nabla_x d$ which can be solved by using the vanishing condition at $\zeta = \infty$, while the projection on $\text{Null}(\mathcal{A}^d)^\perp$ gives the second order equations of $u_0^b \cdot \nabla_x \pi$ and θ_0^b which can be solved by using the vanishing condition at $\zeta = \infty$ and the Robin boundary condition at $\zeta = 0$, i.e. (8.20).

1. Solve $\rho_1^{k,b} + \theta_1^{k,b}$: We first project the system (8.24) on $\text{Null}(\mathcal{A}^d)^\perp$, the u -component of is

$$\partial_\zeta(\rho_1^{k,b} + \theta_1^{k,b}) = 0, \quad \text{hence} \quad \rho_1^{k,b} + \theta_1^{k,b} = 0. \quad (8.26)$$

The ρ -component (or equivalently the θ -component) of the projection on $\text{Null}(\mathcal{A}^d)^\perp$ is

$$-\partial_\zeta(u_1^{k,b} \cdot \nabla_x d) = \text{div}_\pi(u_0^{k,b} \cdot \nabla_x \pi) + \kappa \partial_\zeta^2 \theta_0^{k,b}, \quad (8.27)$$

to solve which we need to first solve $u_0^{k,b} \cdot \nabla_x \pi$ and $\theta_0^{k,b}$. Note that in the derivation of (8.26) and (8.27) the relations (8.6) and (8.7) are used.

2. Solve $[u_0^{k,b}]^{\text{tan}}$: We next project the system (8.24) on $\text{Null}(\mathcal{A}^d)$, the u -component gives the equation for $u_0^{k,b} \cdot \nabla_x \pi^\alpha$:

$$\begin{aligned} (\nu \partial_\zeta^2 - i\lambda_0^k) [u_0^{k,b} \cdot \nabla_x \pi^\alpha] &= 0, \\ [u_0^{k,b} - \frac{\nu}{\chi} \partial_\zeta u_0^{k,b}] (\zeta = 0) \cdot \nabla_x \pi^\alpha &= -u_0^{k,\text{int}}(\pi(x)) \cdot \nabla_x \pi^\alpha, \\ \lim_{\zeta \rightarrow \infty} u_0^{k,b} \cdot \nabla_x \pi^\alpha &= 0. \end{aligned} \quad (8.28)$$

where the boundary condition in the second line of (8.28) follows from the fact that $u_0^{k,b} \cdot \nabla_x \pi^\alpha$ is the tangential components of $u_0^{k,b}$ because $\nabla_x \pi^\alpha$ is tangential to $\partial\Omega$, see the arguments after (4.3). The solution to ODE (8.28) is

$$u_0^{k,b}(\pi(x), \zeta) \cdot \nabla_x \pi^\alpha = \frac{1}{c_\chi \sqrt{\nu-1}} (u_0^{k,\text{int}}(\pi(x)) \cdot \nabla_x \pi^\alpha) \exp \left(-(1 + \tau i) \sqrt{\frac{\lambda_0^k}{2\nu}} \zeta \right), \quad (8.29)$$

where $\tau = +$ or $-$, $c_\chi = -\frac{1+\tau i}{2\chi}\sqrt{2\lambda_0^k}$. We denote the solution (8.29) by

$$u_0^{k,b}(\pi(x), \zeta) \cdot \nabla_x \pi = \tilde{Z}_0^{b,u}(\zeta, U_0^{k,\text{int}}), \quad (8.30)$$

where $\tilde{Z}_0^{b,u}(\zeta, \cdot)$ is a linear function. Note that in the righthand side of (8.29), $U_0^{k,\text{int}}$ should be understood as its value on the boundary, i.e. $U_0^{k,\text{int}}(\pi(x))$.

3. Solve $\theta_0^{k,b}$: The ρ -component (or equivalently the θ -component) of the projection $\text{Null}(\mathcal{A}^d)$ yields the equation for $\theta_0^{k,b}$:

$$\begin{aligned} (\kappa \partial_\zeta^2 - i\lambda_0^k) \theta_0^{k,b} &= 0, \\ [\theta_0^{k,b} - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\zeta \theta_0^{k,b}](\zeta=0) &= -\theta_0^{k,\text{int}}(\pi(x)), \\ \lim_{\zeta \rightarrow \infty} \theta_0^{k,b} &= 0. \end{aligned} \quad (8.31)$$

The solution to (8.31) is

$$\theta_0^{k,b}(\pi(x), \zeta) = \frac{1}{c_\chi \sqrt{\tilde{\kappa}-1}} \theta_0^{k,\text{int}}(\pi(x)) \exp\left(-(1+\tau i)\sqrt{\frac{\lambda_0^k}{2\kappa}}\zeta\right), \quad (8.32)$$

where $\tilde{\kappa} = (\frac{D+2}{D+1})^2 \kappa$. We denote the solution (8.32) by

$$\theta_0^{k,b}(\pi(x), \zeta) = \tilde{Z}_0^{b,\theta}(\zeta, U_0^{k,\text{int}}), \quad (8.33)$$

where $\tilde{Z}_0^{b,\theta}(\zeta, \cdot)$ is a linear function.

4. Solve $u_1^{k,b} \cdot \nabla_x d$: Now the equation (8.27) becomes

$$\partial_\zeta (u_1^{k,b} \cdot \nabla_x d) = -\partial_{\pi^\alpha} (u_0^{k,b} \cdot \nabla_x \pi^\alpha) - i\lambda_0^k \theta_0^{k,b}. \quad (8.34)$$

By integrating the equation (8.34) from ζ to ∞ , it gives

$$u_1^{k,b} \cdot \nabla_x d = \tilde{Z}_1^b(\zeta, U_0^{k,\text{int}}),$$

where $\tilde{Z}_{1,0}^b(\zeta, \cdot)$ is linear. In particular, letting $\zeta = 0$ gives the value of $u_1^{k,b} \cdot \mathbf{n}$ on the boundary $\partial\Omega$:

$$\begin{aligned} -u_1^{k,b} \cdot \mathbf{n} &= \frac{1-\tau i}{\sqrt{2\lambda_0^k}} \left(\text{div}_\pi (u_0^{k,\text{int}} \cdot \nabla_x \pi) \frac{\sqrt{\nu}}{c_\chi \sqrt{\nu-1}} + \tau i \lambda_0^k \theta_0^{k,\text{int}} \frac{\sqrt{\kappa}}{c_\chi \sqrt{\tilde{\kappa}-1}} \right) \\ &= Z_1^b(U_0^{k,\text{int}}) = \tilde{Z}_1^b(0, U_0^{k,\text{int}}), \end{aligned} \quad (8.35)$$

where $Z_1^b(\cdot)$ is a linear function. Consequently, (8.21) gives the boundary value

$$u_1^{k,\text{int}} \cdot \mathbf{n} = Z_1^b(U_0^{k,\text{int}}).$$

Finally we can represent $g_2^{k,b}$ from (8.23):

$$\begin{aligned} g_2^{k,b} &= (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U_2^{k,b} + \partial_\zeta u_1^{k,b} \otimes \nabla_x d : \hat{A} + \partial_\zeta \theta_1^{k,b} \nabla_x d \cdot \hat{B} \\ &\quad + \partial_{\pi^\alpha} u_0^{k,b} \otimes \nabla_x \pi^\alpha : \hat{A} + \partial_{\pi^\alpha} \theta_0^{k,b} \nabla_x \pi^\alpha \cdot \hat{B} \\ &\quad + \mathcal{L}^{-1} \mathcal{P}^\perp \left(\partial_i d \partial_j d \partial_{\zeta\zeta}^2 (u_0^{k,b})_k v_i \hat{A}_{jk} + \partial_i d \partial_j d \partial_{\zeta\zeta}^2 \theta_0^{k,b} v_i \hat{B}_j \right) \\ &= B_0(U_2^b) + B_1(U_1^b) + B_2(U_0^b), \end{aligned} \quad (8.36)$$

where

$$B_2(U_0^b) := \mathcal{L}^{-1} \mathcal{P}^\perp \left((v \cdot \nabla_x d) \partial_\zeta B_1(U_0^b) + (v \cdot \nabla_x \pi^\alpha) \partial_{\pi^\alpha} B_0(U_0^b) \right).$$

Solve $g_1^{k,bb}$: Now we can represent (8.17) as,

$$\begin{aligned} H_1^{bb}(U_0^k) &= h_1^{bb}(U_0^{k,int}) \\ &= L^{\mathcal{R}}(\partial_{\xi} \tilde{Z}_0^{b,u}(0, U_0^k) \nabla_x \pi \otimes n : \hat{A} + \partial_{\xi} \tilde{Z}_0^{b,\theta}(0, U_0^k) n \cdot \hat{B}) \\ &\quad - \frac{\nu}{\chi} v \cdot \partial_{\xi} \tilde{Z}_0^{b,u}(0, U_0^k) \nabla_x \pi + \left(\frac{D+1}{2} - \frac{|v|^2}{2} \right) \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_{\xi} \tilde{Z}_0^{b,\theta}(0, U_0^k), \end{aligned} \quad (8.37)$$

which is completely determined. Thus we can solve $g_1^{k,bb}$ which we denote by

$$g_1^{k,bb}(\pi(x), \xi, v) = K_1(\xi, v, U_0^{k,int}(\pi(x))),$$

where $K_1(\xi, v, \cdot)$ is a linear function.

We summarize that in **Step 2** by considering the order $O(\sqrt{\varepsilon^0})$ in the viscous boundary layer, we determine:

- $\rho_1^{k,b} + \theta_1^{k,b}$;
- $u_0^{k,b} \cdot \nabla_x \pi$ and $\theta_0^{k,b}$, thus $g_0^{k,b}$;
- $u_1^{k,b} \cdot \nabla_x d$ and hence the boundary value of $u_1^{k,b} \cdot n$ when we take $\zeta = 0$, and consequently $u_1^{k,int} \cdot n$ which will be used in **Step 3**;
- expression of $g_2^{k,b}$;
- $g_1^{k,bb}$.

Step 3: Order $O(\sqrt{\varepsilon^1})$ in the interior.

The order $O(\sqrt{\varepsilon^1})$ in the interior part yields

$$\mathcal{L}_3^{k,int} = v \cdot \nabla_x g_1^{k,int} - i\lambda_0^k g_1^{k,int} - i\lambda_1^k g_0^{k,int}, \quad (8.38)$$

and the solvability condition of which is

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k) U_1^{k,int} &= i\lambda_1^k U_0^{k,int}, \quad \text{in } \Omega, \\ u_1^{k,int} \cdot n &= Z_1^b(U_0^{k,int}), \quad \text{on } \partial\Omega. \end{aligned} \quad (8.39)$$

To solve (8.39), we apply Lemme 5.1. The formula (5.20) gives

$$i\lambda_1^k = \int_{\partial\Omega} [u_1^{k,int} \cdot n] \Psi^k d\sigma_x = \int_{\partial\Omega} Z_1^b(U_0^{k,int}) \Psi^k d\sigma_x. \quad (8.40)$$

Note that $\nabla_x \Psi^k = g^{\gamma\beta} \frac{\partial \Psi^k}{\partial \pi^\beta} \frac{\partial}{\partial \pi^\gamma}$, and $\nabla_x \pi^\alpha = g^{\alpha\delta} \frac{\partial}{\partial \pi^\delta}$, we have

$$\begin{aligned} \int_{\partial\Omega} \partial_{\pi^\alpha} (\nabla_x \Psi^k \cdot \nabla_x \pi^\alpha) \Psi^k d\sigma_x &= - \int_{\partial\Omega} g_{\gamma\delta} g^{\alpha\delta} g^{\beta\gamma} \frac{\partial \Psi^k}{\partial \pi^\alpha} \frac{\partial \Psi^k}{\partial \pi^\beta} d\sigma_x \\ &= - \int_{\partial\Omega} |\nabla_\pi \Psi^k|^2 d\sigma_x, \end{aligned}$$

where ∇_π is the tangential gradient on $\partial\Omega$. Thus

$$i\lambda_1^k = \Lambda_1 \int_{\partial\Omega} |\nabla_\pi \Psi^k|^2 d\sigma_x + \Lambda_2 \int_{\partial\Omega} \frac{2}{D+2} (\lambda_0^k)^2 |\Psi^k|^2 d\sigma_x, \quad (8.41)$$

where

$$\begin{aligned} \Lambda_1 &= -\frac{\sqrt{\nu}}{\sqrt{2(\lambda_0^k)^3}} \frac{(2a+1)+\tau i}{(a+1)^2+a^2} \sqrt{\frac{D+2}{D}}, \quad \Lambda_2 = -\frac{\sqrt{\kappa}}{\sqrt{2(\lambda_0^k)^3}} \frac{(2b+1)+\tau i}{(b+1)^2+b^2} \sqrt{\frac{D+2}{D}}, \\ a &= \frac{\sqrt{2\lambda_0^k \nu}}{2\chi}, \quad b = \frac{\sqrt{2\lambda_0^k \kappa}}{2\chi} \frac{D+2}{D+1}. \end{aligned}$$

From the expression (8.41), $i\lambda_1^{\tau,k}$ has an important property (**no matter $\tau = +$ or $-$**):

$$\text{Re}(i\lambda_1^{\tau,k}) < 0. \quad (8.42)$$

Proof. From (8.41), we can only conclude $\operatorname{Re}(i\lambda_1^k) \leq 0$. The strict negativity comes from the following argument. Indeed, assume that $\operatorname{Re}(i\lambda_1^k) = 0$. This would imply that $\nabla_x \Psi^k = 0$ and $\Psi^k = 0$ on the boundary $\partial\Omega$. Hence, extending Ψ^k by 0 outside of Ω and denoting by $\tilde{\Psi}^k$ this extension, we see that $\tilde{\Psi}^k$ is an eigenvector of $-\Delta_x$ on the whole space with compact support which is impossible. Hence (8.42) holds. \square

Remark: The above formula (8.41) and the strict inequality (8.42) is crucial in this paper, because it gives dissipativity, which will be used later in proving the damping of the acoustic waves in the Navier-Stokes limit.

Case 1: If $\frac{D}{D+2}[\lambda_0^k]^2$ is a *simple* eigenvalue of $-\Delta_x$ with Neumann boundary condition, see (5.6). By Lemma 5.1, (8.40) is the only solvability condition under which the system (8.39) can be solved uniquely as

$$U_1^{k,\text{int}} = Z_1^{\text{int}}(U_0^{k,\text{int}}), \quad (8.43)$$

where $Z_1^{\text{int}}(U_0^{k,\text{int}}) \in \operatorname{Null}(\mathcal{A})^\perp$. Note that the system (8.39) is linear and the boundary data Z_1^{bb} is also linear in $U_0^{k,\text{int}}$. So $Z_1^{\text{int}}(U_0^{k,\text{int}})$ also linearly depends on $U_0^{k,\text{int}}$, i.e. $Z_1^{\text{int}}(\cdot)$ is a linear function.

Case 2: If the eigenvalues λ_0^k is *not simple*, an additional compatibility condition is needed, which is given by the formula (5.22):

$$\int_{\partial\Omega} Z_1^{\text{b}}(U_0^{k,\text{int}}) \Psi^l \, d\sigma_x = 0, \quad \text{if } \lambda_0^k = \lambda_0^l \quad \text{and} \quad k \neq l. \quad (8.44)$$

Specifically, this condition reads as

$$\Lambda_1 \int_{\partial\Omega} \nabla_\pi \Psi^k \cdot \nabla_\pi \Psi^l \, d\sigma_x + \Lambda_2 \int_{\partial\Omega} \frac{2}{D+2} (\lambda_0^k)^2 \Psi^k \Psi^l \, d\sigma_x = 0, \quad \text{if } \lambda_0^k = \lambda_0^l \quad \text{and} \quad k \neq l.$$

We can define the quadratic form Q_1 and the symmetric operator L_1 on $H_0(\lambda)$ as

$$Q_1(\Psi^k, \Psi^l) = \int_{\partial\Omega} Z_1^{\text{b}}(U_0^{k,\text{int}}) \Psi^l \, d\sigma_x, \quad (8.45)$$

and

$$L_1 \Psi^k = i\lambda_1^k \Psi^k, \quad (8.46)$$

and the orthogonality condition (8.44) is

$$Q_1(\Psi^k, \Psi^l) = 0, \quad \text{if } \Psi^k, \Psi^l \in H_0(\lambda) \quad \text{and} \quad l \neq k. \quad (8.47)$$

Under these conditions, applying Lemma 5.1, we solve $U_1^{k,\text{int}}$ modulo $\operatorname{Ker}(\mathcal{A} - i\lambda_0^k)$, i.e.

$$U_1^{k,\text{int}} = Z_1^{\text{int}}(U_0^{k,\text{int}}) + P_0 U_1^{k,\text{int}}, \quad (8.48)$$

where $P_0 U_1^{k,\text{int}}$ is defined as

$$P_0 U_1^{k,\text{int}} = \sum_{l \neq k, \lambda_0^l = \lambda_0^k} a_1^{kl} U_0^{l,\text{int}}, \quad (8.49)$$

where $a_1^{kl} = \langle U_1^{k,\text{int}} | U_0^{l,\text{int}} \rangle$ will be determined later. Finally we can represent $g_3^{k,\text{int}}$ as

$$\begin{aligned} g_3^{k,\text{int}} &= (1, v, \frac{|v|^2}{2} - \frac{D}{2}) U_3^{k,\text{int}} + \nabla_x u_1^{k,\text{int}} : \hat{\mathbf{A}}(v) + \nabla_x \theta_1^{k,\text{int}} \cdot \hat{\mathbf{B}}(v) \\ &= I_0(U_3^k) + I_2(P_0 U_1^k) + I_3(U_0^{k,\text{int}}). \end{aligned} \quad (8.50)$$

If λ_0^k is a *simple* eigenvalue, the $P_0 U_1^k$ term vanishes. Thus we finish the **Round 1** in the induction.

Remark: The orthogonality condition (8.44) are given on the eigenfunctions of $-\Delta_x$ with Neumann boundary condition with respect to the eigenvalue $[\lambda_0^k]^2$. Usually, the eigenfunctions

with Neumann boundary conditions are determined up to some constants, so not unique. (8.44) is only used to determine the eigenfunctions corresponding to multiple eigenvalue, and it does not give any new assumption on the geometry of the domain Ω .

8.5. Induction: Round 2. Now we move to the second round in the induction, which also includes three steps by considering terms in the kinetic, viscous boundary layers and interior alternatively.

Step 1: Order $O(\sqrt{\varepsilon}^0)$ in the kinetic boundary layer.

The order $O(\sqrt{\varepsilon}^0)$ of the kinetic boundary layer in the ansatz gives that $g_2^{k,bb}$ satisfies the linear boundary layer equation

$$\begin{aligned} \mathcal{L}^{BL} g_2^{k,bb} &= 0, \quad \text{in } \xi > 0, \\ g_2^{k,bb} &\longrightarrow 0, \quad \text{as } \xi \rightarrow \infty, \end{aligned} \quad (8.51)$$

with boundary condition at $\xi = 0$

$$L^{\mathcal{R}} g_2^{k,bb} = H_2^{k,bb}, \quad \text{on } \xi = 0, \quad v \cdot n > 0, \quad (8.52)$$

where $H_2^{k,bb}$ is of the form:

$$\begin{aligned} H_2^{k,bb} &= -L^{\mathcal{R}} g_2^k + L^{\mathcal{D}}(\tilde{g}_1^k + g_1^{k,bb}) \\ &= \tilde{h}_0^{bb}(U_2^{\text{int}}, U_2^b) + \tilde{h}_1^{bb}(U_1^{\text{int}}, U_1^b) + \tilde{h}_2^{bb}(U_0^{\text{int}}, U_0^b). \end{aligned} \quad (8.53)$$

Here

$$\tilde{h}_2^{bb}(U_0^{k,\text{int}}, U_0^b) = -L^{\mathcal{R}}(I_2(U_0^{\text{int}}) + B_2(U_0^b)) + L^{\mathcal{D}}(B_1(U_0^b) + K_1(U_0^b)). \quad (8.54)$$

Comparing with (8.17), we note that the first two terms of (8.53) are the same as (8.17) replacing by arguments with subscripts higher than 1. In other words,

$$\tilde{h}_0^{bb}(U_2^{\text{int}}, U_2^b) := -2(v \cdot n)(\tilde{u}_2^k \cdot n), \quad (8.55)$$

and

$$\tilde{h}_1^{bb}(U_1^{\text{int}}, U_1^b) := L^{\mathcal{R}}(\partial_\xi u_1^{k,b} \otimes n : \hat{A} + \partial_\xi \theta_1^{k,b} n \cdot \hat{B}) - v \cdot [\tilde{u}_1^k \cdot \nabla_x \pi^\alpha] \nabla_x \pi^\alpha + \left(\frac{D+1}{2} - \frac{|v|^2}{2}\right) \tilde{\theta}_1^k. \quad (8.56)$$

Note that comparing to (8.19), we have used the boundary condition $u_1^k \cdot n = 0$.

Next, the formulas (6.9) and (6.10) give the boundary conditions

$$[u_1^{k,b} - \frac{\nu}{\chi} \partial_\xi u_1^{k,b}]^{\text{tan}} = -[u_1^{k,\text{int}}]^{\text{tan}} + V_1^u(U_0^{k,\text{int}}), \quad (8.57)$$

and

$$\theta_{1,0}^{k,b} - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\xi \theta_1^{k,b} = -\theta_1^{k,\text{int}} + V_1^\theta(U_0^{k,\text{int}}), \quad (8.58)$$

where

$$\begin{aligned} V_1^u(U_0^{k,\text{int}}) &= -\frac{\nu}{\chi} [2d(u_0^{k,\text{int}} \cdot n)]^{\text{tan}} + \left[\int_{v \cdot n > 0} L^{\mathcal{D}} S_1(v \cdot n) v M dv \right]^{\text{tan}}, \\ V_1^\theta(U_0^{k,\text{int}}) &= \frac{\sqrt{2\pi}}{D+1} \int_{v \cdot n > 0} L^{\mathcal{D}} S_1(v \cdot n) |v|^2 M dv, \end{aligned} \quad (8.59)$$

and $S_1 = -(\partial_\xi u_0^{k,b} \otimes n : \hat{A} + \partial_\xi \theta_0^{k,b} n \cdot \hat{B}) + K_1(U_0^{k,\text{int}})$. Here we have used the facts

$$\begin{aligned} &\left[n_i n_j n_l \partial_\xi^2 (u_0^{k,b})_k \int_{\mathbb{R}^D} \mathcal{L}^{-1} \mathcal{P}^\perp(v_i \hat{A}_{jk}) v_l v_m M dv \right]^{\text{tan}} \\ &+ \left[n_i n_j n_l \partial_\xi^2 \theta_0^{k,b} \int_{\mathbb{R}^D} \mathcal{L}^{-1} \mathcal{P}^\perp(v_i \hat{B}_j) v_l v_m M dv \right]^{\text{tan}} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} & n_i n_j n_l \partial_{\zeta}^2 (u_0^{k,b})_k \int_{\mathbb{R}^D} \mathcal{L}^{-1} \mathcal{P}^\perp (v_i \widehat{A}_{jk}) v_l |v|^2 M dv \\ & + n_i n_j n_l \partial_{\zeta}^2 \theta_0^{k,b} \int_{\mathbb{R}^D} \mathcal{L}^{-1} \mathcal{P}^\perp (v_i \widehat{B}_j) v_l |v|^2 M dv \\ & = 0, \end{aligned}$$

Thus, (8.56) is reduced to

$$\begin{aligned} \tilde{h}_1^{\text{bb}}(U_1^{\text{int}}, U_1^{\text{b}}) &= L^{\mathcal{R}}(\partial_{\zeta} u_1^{k,b} \otimes \mathbf{n} : \widehat{\mathbf{A}} + \partial_{\zeta} \theta_1^{k,b} \mathbf{n} \cdot \widehat{\mathbf{B}}) \\ &- \frac{\nu}{\chi} v \cdot [\partial_{\zeta} u_1^{k,b}]^{\text{tan}} - v \cdot V_1^{\text{u}}(U_0^{k,\text{int}}) + \left(\frac{D+1}{2} - \frac{|v|^2}{2}\right) \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_{\zeta} \theta_1^{k,b} + \left(\frac{D+1}{2} - \frac{|v|^2}{2}\right) V_1^{\theta}(U_0^{k,\text{int}}). \end{aligned} \quad (8.60)$$

Furthermore, the formula (6.8) gives the boundary condition on the normal direction:

$$u_2^{k,\text{int}} \cdot \mathbf{n} = -u_2^{k,b} \cdot \mathbf{n}, \quad \text{on } \partial\Omega, \quad (8.61)$$

from which we know $\tilde{h}_0^{\text{bb}}(U_2^{\text{int}}, U_2^{\text{b}}) = 0$. Thus from (8.53), to solve $g_2^{k,\text{bb}}$ we don't need know $g_2^{k,\text{int}}$ and $g_2^{k,b}$, although formally it does. Thus, once we solve $u_1^{k,b} \cdot \nabla_x \pi$ and $\theta_1^{k,b}$, we can solve $g_2^{k,\text{bb}}$.

Step 2: Order $O(\sqrt{\varepsilon}^1)$ in the viscous boundary layer.

The equations of $u_1^{k,b} \cdot \nabla_x \pi$ and $\theta_1^{k,b}$ can be found by analyzing the order $O(\sqrt{\varepsilon}^1)$ of the viscous boundary layer in (7.5) which gives

$$\mathcal{L} g_3^{k,b} = (v \cdot \nabla_x d) \partial_{\zeta} g_2^{k,b} + (v \cdot \nabla_x \pi^{\alpha}) \partial_{\pi^{\alpha}} g_1^{k,b} - i \lambda_0^k g_1^{k,b} - i \lambda_1^k g_0^{k,b}. \quad (8.62)$$

The solvability condition for (8.62) yields that

$$-\mathcal{A}^d U_2^{k,b} = (\mathcal{A}^{\pi} + \mathcal{D}^d - i \lambda_0^k) U_1^{k,b} + (\mathcal{F}_1 - i \lambda_1^k) U_0^{k,b}, \quad (8.63)$$

where the linear operator $\mathcal{F}_1(U_0^{k,b}) = (\mathcal{F}_1^{\rho}, \mathcal{F}_1^{\text{u}}, \mathcal{F}_1^{\theta})^{\top}(U_0^{k,b})$ is defined as

$$\mathcal{F}_1(U_0^{k,b}) \cdot (1, v, \frac{|v|^2}{2} - \frac{D}{2}) := \mathcal{P} \left\{ v \cdot \nabla_x d \partial_{\zeta} B_2(U_0^{k,b}) + v \cdot \nabla_x \pi \partial_{\pi} B_1(U_0^{k,b}) \right\}.$$

The precise form of $\mathcal{F}_1(U_0^{k,b})$ is tedious and not easy to represent explicitly, however, is not of great importance for the later analysis. The ρ component vanishes, while the u and θ components are linear functions of third order ζ derivatives of $u_0^{k,b}$ and $\theta_0^{k,b}$ respectively.

Similar as in solving (8.24), we derive the ODE satisfied by $\rho_2^{k,b} + \theta_2^{k,b}$ from the u -component of the projection of (8.63) on $\text{Null}(\mathcal{A}^d)^{\perp}$:

$$-\partial_{\zeta}(\rho_2^{k,b} + \theta_2^{k,b}) = \left\{ \nu(2 - \frac{2}{D}) \partial_{\zeta}^2 - i \lambda_0^k \right\} [u_1^{k,b} \cdot \nabla_x d] + \mathcal{F}_1^{\text{u}}(U_0^{k,b}) \cdot \nabla_x d. \quad (8.64)$$

Note that both $u_1^{k,b} \cdot \nabla_x d$ and $u_0^{k,b} \cdot \nabla_x \pi^{\alpha}$ (which is included in $\mathcal{F}_1^{\text{u}}(U_0^{k,b}) \cdot \nabla_x d$) are known from the last round and linear in $U_0^{k,\text{int}}$, so the righthand side of (8.64) is known and a linear function of $U_0^{k,\text{int}}$. Integrating (8.64) from ζ to ∞ gives

$$\rho_2^{k,b} + \theta_2^{k,b} = Y_2^{\text{b}}(\zeta, U_0^{k,\text{int}}), \quad (8.65)$$

where $Y_2^{\text{b}}(\zeta, \cdot)$ is a linear function. We can also derive the ODEs satisfied by $u_1^{k,b} \cdot \nabla_x \pi$ and $\theta_1^{k,b}$:

$$\left\{ \nu \partial_{\zeta}^2 - i \lambda_0^k \right\} [u_1^{k,b} \cdot \nabla_x \pi] = i \lambda_1^k [u_0^{k,b} \cdot \nabla_x \pi] - \mathcal{F}_1^{\text{u}}(U_0^{k,b}) \cdot \nabla_x \pi, \quad (8.66)$$

and

$$\left\{ \kappa \partial_{\zeta}^2 - i \lambda_0^k \right\} \theta_1^{k,b} = i \lambda_1^k \theta_0^{k,b} - \left(\frac{2}{D+2} \mathcal{F}_1^{\rho} - \frac{D}{D+2} \mathcal{F}_1^{\theta} \right) (U_0^{k,b}), \quad (8.67)$$

with boundary conditions (8.57) and (8.58) respectively. Because of the linearity of the above equations, we can solve (8.66) and (8.67) as

$$\begin{aligned} u_1^{k,b} \cdot \nabla_x \pi &= \tilde{Z}_0^{b,u}(\zeta, P_0 U_1^{k,int}) + \tilde{Z}_1^{b,u}(\zeta, U_0^{k,int}), \\ \theta_1^{k,b} &= \tilde{Z}_0^{b,\theta}(\zeta, P_0 U_1^{k,int}) + \tilde{Z}_1^{b,\theta}(\zeta, U_0^{k,int}), \end{aligned} \quad (8.68)$$

recalling $\tilde{Z}_0^{b,u}$ and $\tilde{Z}_0^{b,\theta}$ are defined in (8.29) and (8.32) respectively, and $\tilde{Z}_1^{b,u}$ is the solution of

$$\begin{aligned} (\nu \partial_\zeta^2 - i\lambda_0^k)u &= i\lambda_1^k[u_0^{k,b} \cdot \nabla_x \pi] - \mathcal{F}_1^u(U_0^{k,b}) \cdot \nabla_x \pi, \\ [u - \frac{\nu}{\chi} \partial_\zeta u](\zeta = 0) &= -Z_1^{\text{int}}(U_0^{k,int})_u \cdot \nabla_x \pi + V_1^u(U_0^{k,int}) \cdot \nabla_x \pi, \\ \lim_{\zeta \rightarrow \infty} u &= 0, \end{aligned}$$

and $\tilde{Z}_1^{b,\theta}$ is the solution of

$$\begin{aligned} (\nu \partial_\zeta^2 - i\lambda_0^k)\theta &= i\lambda_1^k\theta_0^{k,b} - \mathcal{F}_1^\theta(U_0^{k,b}), \\ [\theta - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_\zeta \theta](\zeta = 0) &= -Z_1^{\text{int}}(U_0^{k,int})_\theta + V_1^\theta(U_0^{k,int}), \\ \lim_{\zeta \rightarrow \infty} \theta &= 0, \end{aligned}$$

where $Z_1^{\text{int}}(U_0^{k,int})_u$ and $Z_1^{\text{int}}(U_0^{k,int})_\theta$ denote the u and θ components of $Z_1^{\text{int}}(U_0^{k,int})$ respectively.

From the ρ -component of the projection of (8.63) on $\text{Null}(\mathcal{A}^d)^\perp$ we can also derive the equation for $u_2^{k,b} \cdot \nabla_x d$:

$$-\partial_\zeta[u_2^{k,b} \cdot \nabla_x d] = \text{div}_\pi[u_1^{k,b} \cdot \nabla_x \pi] + i\lambda_0^k\theta_1^{k,b} + i\lambda_{1,0}^k\theta_0^{k,b}.$$

Integrating from ζ to ∞ , we can solve $u_2^{k,b} \cdot \nabla_x d = \tilde{Z}_1^b(\zeta, P_0 U_1^{k,int}) + \tilde{Z}_2^b(\zeta, U_0^{k,int})$, in particular, by taking $\zeta = 0$

$$-u_2^{k,b} \cdot n = Z_1^b(P_0 U_1^{k,int}) + Z_2^b(U_0^{k,int}) = u_2^{k,int} \cdot n, \quad (8.69)$$

where the second equality followed from (8.61).

Finally, $g_3^{k,b}$ can be represented as

$$\begin{aligned} g_3^{k,b} &= (1, v, \frac{|v|^2}{2} - \frac{D}{2})U_3^{k,b} + \partial_\zeta u_2^{k,b} \otimes \nabla_x d : \hat{A} + \partial_\zeta \theta_2^{k,b} \nabla_x d \cdot \hat{B} \\ &\quad + \partial_{\pi^\alpha} u_1^{k,b} \otimes \nabla_x \pi^\alpha : \hat{A} + \partial_{\pi^\alpha} \theta_1^{k,b} \nabla_x \pi^\alpha \cdot \hat{B} \\ &\quad + \mathcal{L}^{-1} \mathcal{P}^\perp \left(\partial_i d \partial_j d \partial_{\zeta\zeta}^2 (u_1^{k,b})_k v_i \hat{A}_{jk} + \partial_i d \partial_j d \partial_{\zeta\zeta}^2 \theta_1^{k,b} v_i \hat{B}_j \right) \\ &\quad + \mathcal{L}^{-1} \mathcal{P}^\perp \left((\partial_i d \partial_j \pi^\alpha + \partial_j d \partial_i \pi^\alpha) (\partial_{\zeta\pi^\alpha}^2 (u_0^{k,b})_k v_i \hat{A}_{jk} + \partial_{\zeta\pi^\alpha}^2 \theta_0^{k,b} v_i \hat{B}_j) \right) \\ &\quad + \mathcal{L}^{-1} \mathcal{P}^\perp \left(\partial_i d \partial_j d \partial_k d [\partial_{\zeta\zeta\zeta}^3 (u_0^{k,b})_l v_i \mathcal{L}^{-1} \mathcal{P}^\perp(v_j \hat{A}_{kl}) + \partial_{\zeta\zeta\zeta}^3 \theta_{0,0}^{k,b} v_i \mathcal{L}^{-1} \mathcal{P}^\perp(v_j \hat{B}_k)] \right) \\ &\quad - i\lambda_0^k \left(\partial_\zeta u_0^{k,b} \otimes \nabla_x d : \mathcal{L}^{-1} \hat{A} + \partial_\zeta \theta_0^{k,b} \nabla_x d \cdot \mathcal{L}^{-1} \hat{B} \right) \\ &= B_0(U_3^b) + B_1(U_2^b) + B_2(U_1^b) + B_3(U_0^b), \end{aligned}$$

where $B_3(v, U_0^b)$ can be represented as

$$\begin{aligned} &B_3(v, U_0^b) \\ &= \mathcal{L}^{-1} \mathcal{P}^\perp \left((v \cdot \nabla_x d) \partial_\zeta B_2(v, U_0^b) + [(v \cdot \nabla_x \pi^\alpha) \partial_{\pi^\alpha} - i\lambda_0^k] B_1(v, U_0^b) \right). \end{aligned}$$

After $g_1^{k,b}$ is solved (modulo $P_0 U_1^{k,int}$), go back to the linear kinetic boundary layer equations (8.51)-(8.52) of $g_2^{k,bb}$. Straightforward calculations by regrouping terms show that

$$H_2^{k,bb} = h_1^{bb}(P_0 U_1^k) + h_2^{bb}(U_0^{k,int}),$$

where $h_2^{\text{bb}}(\cdot)$ is linear and whose detailed expression we omit here. Using the linearity of the kinetic boundary layer equation and the boundary conditions, it is easy to solve that

$$g_2^{k,\text{bb}} = K_1(\xi, v, P_0 U_1^{k,\text{int}}) + K_2(\xi, v, U_0^{k,\text{int}}),$$

where $K_2(\xi, v, U_0^{k,\text{int}})$ is the solution of the kinetic boundary layer equations (8.51)-(8.52) with the boundary condition

$$(\gamma_+ - L\gamma_-)K_2 = h_2^{\text{bb}}(U_0^{k,\text{int}}), \quad \text{on } \xi = 0, \quad v \cdot \mathbf{n} > 0.$$

It is obvious that $K_2(\xi, v, \cdot)$ is linear.

Step 3: Order $O(\sqrt{\varepsilon^2})$ in the interior.

The order $O(\varepsilon)$ in the interior part of (7.5) yields

$$\mathcal{L}g_4^{k,\text{int}} = v \cdot \nabla_x g_2^{k,\text{int}} - i\lambda_0^k g_2^{k,\text{int}} - i\lambda_1^k g_1^{k,\text{int}} - i\lambda_2^k g_0^{k,\text{int}}, \quad (8.70)$$

the solvability condition of which is

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k)U_2^{k,\text{int}} &= i\lambda_1^k U_1^{k,\text{int}} + (i\lambda_2^k - \mathcal{D})U_0^{k,\text{int}} \quad \text{in } \Omega, \\ \mathbf{u}_2^{k,\text{int}} \cdot \mathbf{n} &= Z_1^{\text{b}}(P_0 U_1^{k,\text{int}}) + Z_2^{\text{b}}(U_0^{k,\text{int}}) \quad \text{on } \partial\Omega, \end{aligned} \quad (8.71)$$

where \mathcal{D} is defined as

$$\mathcal{D}U = \begin{pmatrix} 0 \\ \nu \operatorname{div}_x(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top - \frac{2}{\mathcal{D}} \operatorname{div}_x \mathbf{u}) \\ \frac{\mathcal{D}+2}{\mathcal{D}} \kappa \Delta_x \theta \end{pmatrix}.$$

Remark: To apply Lemma 5.1, we require the following orthogonality condition for $U_m^{k,\text{int}}$:

$$\langle U_m^{k,\text{int}} | U_0^{l,\text{int}} \rangle = 0, \quad \text{for all } m \neq 0 \text{ or } k \neq l. \quad (8.72)$$

To solve (8.71), first we apply the formula (5.20) of Lemma 5.1 to deduce

$$\begin{aligned} i\lambda_2^k &= \int_{\partial\Omega} [\mathbf{u}_2^{k,\text{int}} \cdot \mathbf{n}] \Psi^k d\sigma_x \\ &= \int_{\partial\Omega} Z_2^{\text{b}}(U_0^{k,\text{int}}) \Psi^k d\sigma_x + \int_{\partial\Omega} Z_1^{\text{b}}(P U_0^{k,\text{int}}) \Psi^k d\sigma_x + \langle \mathcal{D}U_0^{k,\text{int}} | U_0^{k,\text{int}} \rangle, \\ &= \int_{\partial\Omega} Z_2^{\text{b}}(U_0^{k,\text{int}}) \Psi^k d\sigma_x + \langle \mathcal{D}U_0^{k,\text{int}} | U_0^{k,\text{int}} \rangle. \end{aligned} \quad (8.73)$$

Case 1: If λ_0^k is a *simple* eigenvalue, then $P U_0^{k,\text{int}} = 0$, thus $i\lambda_1^k$ is given by (8.73).

Otherwise, λ_0^k is not a *simple* eigenvalue, note that

$$\begin{aligned} \int_{\partial\Omega} Z_1^{\text{b}}(P U_0^{k,\text{int}}) \Psi^k d\sigma_x &= \sum_{l \neq k, \lambda_0^l = \lambda_0^k} a_1^{kl} \int_{\partial\Omega} Z_1^{\text{b}}(U_0^{l,\text{int}}) \Psi^k d\sigma_x \\ &= 0, \end{aligned} \quad (8.74)$$

because of the orthogonality condition (8.45) and (8.47). The above two identities illustrate that no matter λ_0^k is simple or not, $i\lambda_2^k$ is completely determined, which is given by (8.73).

When λ_0^k is not simple, the compatibility condition (5.22) is needed, which gives

$$i\lambda_1^l a_1^{kl} + \int_{\partial\Omega} Z_2^{\text{b}}(U_0^{k,\text{int}}) \Psi^l d\sigma_x = i\lambda_1^k a_1^{kl} \quad \text{if } \lambda_0^k = \lambda_0^l \text{ and } k \neq l. \quad (8.75)$$

Case 2: If λ_0^k is not a *simple* eigenvalue, but $i\lambda_1^k$ is a *simple* eigenvalue of L_1 which is defined in (8.46), then for all $l \neq k$ with $i\lambda_0^l = i\lambda_0^k$, we have $i\lambda_1^l \neq i\lambda_1^k$. For this case a_1^{kl} can be solved from (8.75) as

$$a_1^{kl} = \frac{1}{i\lambda_1^k - i\lambda_1^l} \int_{\partial\Omega} Z_2^{\text{b}}(U_0^{k,\text{int}}) \Psi^l d\sigma_x. \quad (8.76)$$

Thus $P_0(U_1^{k,\text{int}})$ is completely determined, and no additional conditions on $H_0(\lambda)$ rather than (8.44), or equivalently (8.47) is needed. However,

Case 3: If λ_0^k is not a *simple* eigenvalue, and $i\lambda_1^k$ is also not a *simple* eigenvalue of L_1 , we need more orthogonality condition on

$$H_1 = H(\lambda_1) = \{\Psi \in H_0 : L_1\Psi = i\lambda_1^k\Psi\}.$$

This orthogonality condition comes from (8.75):

$$\int_{\partial\Omega} Z_2^b(U_0^{k,\text{int}})\Psi^l d\sigma_x = 0, \quad \text{if } l \neq k \quad \lambda_0^l = \lambda_0^k \quad \lambda_1^l = \lambda_1^k. \quad (8.77)$$

We can define a quadratic form Q_2 and the symmetric operator L_2 on $H_1(\lambda_1)$ as

$$Q_2(\Psi^k, \Psi^l) = \int_{\partial\Omega} Z_2^b(U_0^{k,\text{int}})\Psi^l d\sigma_x + \langle \mathcal{D}U_0^{k,\text{int}} | U_0^{l,\text{int}} \rangle, \quad (8.78)$$

and $L_2\Psi^k = i\lambda_2^k\Psi^k$, which satisfies that

$$Q_2(\Psi^k, \Psi^l) = \int_{\Omega} L_2(\Psi^k)\Psi^l dx.$$

Be these definitions, we have $i\lambda_2^k = Q_2(\Psi^k, \Psi^k)$, and the condition (8.77) is

$$Q_2(\Psi^k, \Psi^l) = 0, \quad \text{if } \Psi^k, \Psi^l \in H_1(\lambda_1) \quad \text{and } l \neq k.$$

Under these conditions, the equation (8.71) can be solved in the following way: Let $U_2^{k,\text{int}} = U^1 + U^2$, where U^1 satisfies the equation

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k)U^1 &= i\lambda_1^k P_0 U_1^{k,\text{int}}, \\ u^1 \cdot n &= Z_1^b(P_0 U_1^{k,\text{int}}), \end{aligned}$$

whose solution in $\text{Ker}(\mathcal{A} - i\lambda_0^k)^\perp$ is $Z_1^{\text{int}}(P_0 U_1^{k,\text{int}})$, and U^2 satisfies the equation

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k)U^2 &= i\lambda_1^k Z_1^{\text{int}}(U_0^{k,\text{int}}) + (i\lambda_2^k - \mathcal{D})U_0^{k,\text{int}}, \\ u^2 \cdot n &= Z_2^b(U_0^{k,\text{int}}), \end{aligned}$$

whose solution in $\text{Ker}(\mathcal{A} - i\lambda_0^k)^\perp$ is *completely* determined, and is denoted by $Z_2^{\text{int}}(U_0^{k,\text{int}})$. In summary, the equation (8.71) is

$$U_2^{k,\text{int}} = P_0 U_2^{k,\text{int}} + Z_1^{\text{int}}(P_0 U_1^k) + Z_2^{\text{int}}(U_0^{k,\text{int}}),$$

where $P_0(U_1^{k,\text{int}}) = (P_1 + P_1^\perp)(U_1^{k,\text{int}})$ in which $P_1^\perp U_1^{k,\text{int}}$ is already completely determined in (8.76) and $P_1 U_1^{k,\text{int}}$ will be determined later, and $P_0 U_2^{k,\text{int}}$ is defined the same as in (8.49), i.e.

$$P_0 U_2^{k,\text{int}} = \sum_{l \neq k, \lambda_0^l = \lambda_0^k} a_2^{kl} U_0^{l,\text{int}},$$

where $a_2^{kl} = \langle U_2^{k,\text{int}} | U_0^{l,\text{int}} \rangle$ will be determined later. Finally we can represent $g_4^{k,\text{int}}$ as

$$\begin{aligned} g_4^{k,\text{int}} &= I_0(U_4^k) + I_2(U_2^k) + I_4(U_0^{k,\text{int}}) \\ &:= (1, v, \frac{|v|^2}{2} - \frac{D}{2})U_4^{k,\text{int}} + \nabla_x u_2^{k,\text{int}} \cdot \widehat{A}(v) + \nabla_x \theta_2^{k,\text{int}} \cdot \widehat{B}(v) \\ &\quad + \mathcal{L}^{-1} \mathcal{P}^\perp \left(\partial_{x_i x_j}^2 (u_0^{k,\text{int}})_k v_i \widehat{A}_{jk} + \partial_{x_i x_j}^2 \theta_0^{k,\text{int}} v_i \widehat{B}_j \right) \\ &\quad - i\lambda_0^k \left(\nabla_x u_0^{k,\text{int}} : \mathcal{L}^{-1} \widehat{A} + \nabla_x \theta_0^{k,\text{int}} \cdot \mathcal{L}^{-1} \widehat{B} \right), \end{aligned} \quad (8.79)$$

where $I_4(U_0^{k,\text{int}}) = \mathcal{L}^{-1} \mathcal{P}^\perp \left\{ (v \cdot \nabla_x - i\lambda_0^k) I_2(U_0^{k,\text{int}}) \right\}$. Thus we conclude the **Round 2** in the induction.

8.6. General case: Induction hypothesis. For $m \geq 3$, we assume that we have finished the $(m-1)$ -th round, i.e. used the information from the kinetic boundary layer, the viscous boundary and the interior till the order $O(\sqrt{\varepsilon}^{m-3})$, $O(\sqrt{\varepsilon}^{m-2})$ and $O(\sqrt{\varepsilon}^{m-1})$ respectively. Before we solve the next round, we write down the hypothesis that summarizes what we were able to construct till now. We write down this in following 10 statements that we need to check for the m -th round.

(\mathbf{P}_{m-1}^1) : For $2 \leq j \leq m-1$, $\rho_j^{k,b} + \theta_j^{k,b} = \sum_{h=2}^j Y_h^b(\zeta, P_0 U_{j-h}^{k,\text{int}})$; For $j = 0, 1$, $\rho_j^{k,b} + \theta_j^{k,b} = 0$.

(\mathbf{P}_{m-1}^2) : For $0 \leq j \leq m-2$, $u_j^{k,b} \cdot \nabla_x \pi = \sum_{h=0}^j \tilde{Z}_h^{b,u}(\zeta, P_0 U_{j-h}^{k,\text{int}})$;

(\mathbf{P}_{m-1}^3) : For $0 \leq j \leq m-2$, $\theta_j^{k,b} = \sum_{h=0}^j \tilde{Z}_h^{b,\theta}(\zeta, P_0 U_{j-h}^{k,\text{int}})$;

(\mathbf{P}_{m-1}^4) : For $1 \leq j \leq m-1$, $u_j^{k,b} \cdot \nabla_x d = \sum_{h=1}^j \tilde{Z}_h^b(\zeta, P_0 U_{j-h}^{k,\text{int}})$. Taking $\zeta = 0$, we deduce that on the boundary we have $-u_j^{k,b} \cdot n = \sum_{h=1}^j Z_h^b(P_0 U_{j-h}^{k,\text{int}})$.

(\mathbf{P}_{m-1}^5) : For $0 \leq j \leq m$, $g_j^{k,b} = \sum_{h=0}^j B_h(U_{j-h}^{\text{int}})$, where B_h for $h \geq 0$ is defined iteratively starting from $B_0(U^b) = (1, v, \frac{|v|^2}{2} - \frac{D}{2})U^b$:

$$B_h(U^b) = \mathcal{L}^{-1} \mathcal{P}^\perp \{v \cdot \nabla_x d \partial_\xi B_{h-1}(U^b) + v \cdot \nabla_x \pi \partial_\pi B_{h-1}(U^b) - \sum_{l=0}^{h-3} i \lambda_l^k B_{h-2-l}(U^b)\}. \quad (8.80)$$

(\mathbf{P}_{m-1}^6) : For $1 \leq j \leq m-1$,

$$g_j^{k,bb} = \sum_{h=1}^j K_h(v, \xi, P_0(U_{j-h}^{k,\text{int}})), \quad (8.81)$$

where the linear operator $K_h(v, \xi, U_0^{k,\text{int}})$ is the solution to the linear kinetic boundary layer equation (6.3)-(6.4) with the source term $s_h^{bb}(U_0^{k,\text{int}})$ and the boundary source term $h_h^{bb}(U_0^{k,\text{int}})$.

(\mathbf{P}_{m-1}^7) : For $1 \leq j \leq m-1$, $i \lambda_j^k = Q_j(\Psi^k, \Psi^k)$, where the quadratic form Q_1 and Q_2 are defined in (8.45) and (8.78) respectively, and Q_j for $3 \leq j \leq m-1$ is defined as

$$Q_j(\Psi^k, \Psi^l) = \sqrt{\frac{D+2}{2D}} \int_{\partial\Omega} \{Z_j^b + V_j^n\}(U_0^{k,\text{int}}) \Psi^l d\sigma_x + \sum_{h=2}^j \langle \mathcal{G}_h(Z_{j-h}^{\text{int}}(U_0^{k,\text{int}})) | U_0^{l,\text{int}} \rangle. \quad (8.82)$$

Note \mathcal{G}_h is defined in (8.110).

(\mathbf{P}_{m-1}^8) : For $0 \leq j \leq m-1$, $U_j^{k,\text{int}} = P_0 U_j^{k,\text{int}} + \sum_{h=1}^j Z_h^{\text{int}}(P_0 U_{j-h}^{k,\text{int}})$;

(\mathbf{P}_{m-1}^9) : For $0 \leq j \leq m+1$, $g_j^{k,\text{int}} = I_0(U_j^{\text{int}}) + I_2(U_{j-2}^{\text{int}}) + \sum_{h=4}^j I_h(U_{j-h}^{\text{int}})$, where I_h for $h \geq 0$ is defined iteratively starting from $I_0(U^{\text{int}}) = (1, v, \frac{|v|^2}{2} - \frac{D}{2})U^{\text{int}}$, $I_1 = 0$:

$$I_h(U^{\text{int}}) = \mathcal{L}^{-1} \mathcal{P}^\perp \{v \cdot \nabla_x I_{h-2}(U^{\text{int}}) - \sum_{l=0}^{h-4} i \lambda_l^k I_{h-2-m}(U^{\text{int}})\}. \quad (8.83)$$

(\mathbf{P}_{m-1}^{10}) : The last assumption to check deals with the number of orthogonality conditions needed and specifies what is already determined and what is still not determined in the construction. We distinguish between m cases:

Case 1: $i \lambda_h^k$ is a simple eigenvalue of L_h for $0 \leq h \leq m-2$. No orthogonality condition is needed, and every term in the expansion is fully determined;

Case j ($2 \leq j \leq m$): $i \lambda_h^k$ is a multiple eigenvalue of L_h for $0 \leq h \leq j-2$, but $i \lambda_{j-1}^k$ a simple eigenvalue of L_{j-1} . (Note: the **case m** means that all the eigenvalues $i \lambda_h^k$ for $0 \leq h \leq m-2$ are multiple eigenvalues.)

- We need the orthogonality conditions: For each $0 \leq h \leq j-2$,

$$Q_{h+1}(\Psi^k, \Psi^l) = 0, \quad \text{for } \Psi^k, \Psi^l \in H_0 \cap \dots \cap H_h, \quad (8.84)$$

where for $h \geq 1$, the space $H_h = H_h(\lambda_h) = \{\Psi \in H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1}) : L_h \Psi = i\lambda_h \Psi\}$.

- For $1 \leq h \leq m-j$, $U_h^{k,\text{int}}$ are completely determined. (for the **case m**, no term is completely determined.)
- For $m-j+1 \leq h \leq m-1$, $(P_0^\perp + \cdots + P_{m-1-h}^\perp)U_h^{k,\text{int}}$ are determined.
- For $m-j+1 \leq h \leq m-1$, $P_{m-1-h}U_h^{k,\text{int}}$ are not determined,

where P_{h-1} is the orthogonal projection on $H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1})$, and $P_{h-1} = P_h + P_h^\perp$, where P_h^\perp is the orthogonal projection on $H_1(\lambda_1) \cap \cdots \cap H_{h-1}(\lambda_{h-1}) \cap H_h^\perp(\lambda_h)$.

Remark: Regarding the condition (8.84), actually we have a stronger orthogonality property which is actually equivalent to (8.84), namely : for each $0 \leq h \leq j-2$,

$$Q_{h+1}(\Psi_k, \Psi_l) = 0, \quad \text{for } l \neq k, \quad \Psi_k, \Psi_l \in H_0. \quad (8.85)$$

Indeed, we just need to use that the L_h leave stable the spaces H_h . Of course, we have to define L_h over the whole space H_0 even if the eigenvalue is simple, but in this case we just take it to be the identity.

In the next subsection, we are going to prove the 10 hypotheses $(\mathbf{P}_m^1) - (\mathbf{P}_m^{10})$ assuming \mathbf{P}_{i-1} for $i \leq m$.

8.7. Induction: Round m. For $m \geq 3$, we assume that we have finished round $m-1$ in the induction process. For the round m , as before it includes three steps by considering terms in the kinetic, viscous boundary layers and interior alternatively.

Step 1: Order $O(\sqrt{\varepsilon}^{m-2})$ in the kinetic boundary layer.

The order $O(\sqrt{\varepsilon}^{m-2})$ of the kinetic boundary layer in the ansatz gives that $g_m^{k,\text{bb}}$ satisfies the linear boundary layer equation

$$\begin{aligned} \mathcal{L}^{BL} g_m^{k,\text{bb}} &= S_m^{k,\text{bb}}, \quad \text{in } \xi > 0, \\ g_m^{k,\text{bb}} &\longrightarrow 0, \quad \text{as } \xi \rightarrow \infty, \end{aligned} \quad (8.86)$$

with boundary condition at $\xi = 0$

$$L^{\mathcal{R}} g_m^{k,\text{bb}} = H_m^{k,\text{bb}}, \quad \text{on } \xi = 0, \quad v \cdot n > 0, \quad (8.87)$$

The source term

$$\begin{aligned} S_m^{k,\text{bb}} &= \left\{ v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} - i\lambda_0^k \right\} g_{m-2}^{k,\text{bb}} - \sum_{j=1}^{m-3} i\lambda_j^k g_{m-2-j}^{k,\text{bb}} \\ &= \sum_{j=3}^m s_j^{\text{bb}}(P_0 U_{m-j}^{k,\text{int}}), \end{aligned}$$

where for $3 \leq j \leq m-3$,

$$s_j^{\text{bb}}(U^{\text{int}}) = \left\{ v \cdot \nabla_x \pi \partial_\pi - i\lambda_0^k \right\} K_{j-2}(U^{\text{int}}) - \sum_{h=1}^{j-3} i\lambda_h^k K_{j-2-h}(U^{\text{int}}), \quad (8.88)$$

recalling the functions $K_j(U^{\text{int}})$ are defined through (8.81). The boundary source term is

$$\begin{aligned} H_m^{k,\text{bb}} &= -L^{\mathcal{R}} \tilde{g}_m^k + L^{\mathcal{D}}(\tilde{g}_{m-1}^k + g_{m-1}^{k,\text{bb}}) \\ &= \sum_{j=0}^m \tilde{h}_j^{\text{bb}}(\tilde{U}_{m-j}^k) = \sum_{j=1}^m h_j^{\text{bb}}(P_0 U_{m-j}^{k,\text{int}}). \end{aligned} \quad (8.89)$$

Here

$$\begin{aligned} \tilde{h}_j^{\text{bb}}(\tilde{U}_0^k) = & -L^{\mathcal{R}} \left\{ I_j(U_0^{k,\text{int}}) + B_j(U_0^{k,\text{b}}) \right\} \\ & + L^{\mathcal{D}} \left\{ I_{j-1}(U_0^{k,\text{int}}) + B_{j-1}(U_0^{k,\text{b}}) + K_{j-1}(U_0^{k,\text{int}}) \right\}. \end{aligned}$$

The definition of $h_j^{\text{bb}}(P_0 U_{m-j}^{k,\text{int}})$ is the following: represent all I_0, \dots, I_m and B_0, \dots, B_m in terms of $U_0^{k,\text{int}}, \dots, P_0 U_{m-1}^{k,\text{int}}$, and collect the corresponding terms in $\sum_{j=0}^m \tilde{h}_j^{\text{bb}}(\tilde{U}_{m-j}^k)$, which defines $h_j^{\text{bb}}(P_0 U_{m-j}^{k,\text{int}})$ for $j = 1, \dots, m-1$. Note that there is no $U_m^{k,\text{int}}$ term, since it *formally* only appears in the term $-L^{\mathcal{R}}(I_0(U_m^{\text{int}}) + B_0(U_m^{\text{b}})) = -2(v \cdot n)(u_m^{k,\text{int}} + u_m^{k,\text{b}}) \cdot n$, which only depends on $U_0^{k,\text{int}}, P_0 U_1^{k,\text{int}}, \dots, P_0 U_{m-1}^{k,\text{int}}$, see (\mathbf{P}_{m-1}^4) .

The formulas (6.9) and (6.10) give the boundary conditions

$$[u_{m-1}^{k,\text{b}} - \frac{\nu}{\chi} \partial_\xi u_{m-1}^{k,\text{b}}]^{\text{tan}} + [u_{m-1}^{k,\text{int}}]^{\text{tan}} = \sum_{j=1}^{m-1} V_j^{\text{u}}(P_0 U_{m-1-j}^{\text{int}}), \quad (8.90)$$

and

$$\theta_{m-1}^{k,\text{b}} - \frac{\text{D}+2}{\text{D}+1} \frac{\kappa}{\chi} \partial_\xi \theta_{m-1}^{k,\text{b}} + \theta_{m-1}^{k,\text{int}} = \sum_{j=1}^{m-1} V_j^\theta(P_0 U_{m-1-j}^{\text{int}}), \quad (8.91)$$

where

$$\begin{aligned} \sum_{j=1}^{m-1} V_j^{\text{u}}(P_0 U_{m-1-j}^{k,\text{int}}) = & -\frac{\nu}{\chi} [2d(u_{m-2}^{k,\text{int}}) \cdot n]^{\text{tan}} - \frac{\nu}{\chi} \nabla_\pi [u_{m-2}^{k,\text{b}} \cdot n] \\ & + \sum_{j=1}^{m-1} \int_{v \cdot n > 0} \left[L^{\mathcal{D}} \left\{ B_j(U_{m-1-j}^{k,\text{b}}) + K_j(P_0 U_{m-1-j}^{k,\text{int}}) \right\} (v \cdot n) v \right]^{\text{tan}} M \, dv \\ & - \frac{1}{\chi} \left\langle (v \cdot n) v \left[\sum_{j=4}^m I_j(U_{m-j}^{k,\text{int}}) + \sum_{j=3}^m B_j(U_{m-j}^{k,\text{b}}) \right] \right\rangle^{\text{tan}} + \frac{1}{\chi} \int_0^\infty \langle v S_m^{k,\text{bb}} \rangle^{\text{tan}} d\xi, \end{aligned} \quad (8.92)$$

and

$$\begin{aligned} \sum_{j=1}^{m-1} V_j^\theta(P_0 U_{m-1-j}^{k,\text{int}}) = & -\frac{\text{D}+2}{\text{D}+1} \frac{\kappa}{\chi} \partial_n \theta_{m-2}^{k,\text{int}} + \frac{\sqrt{2\pi}}{2(\text{D}+1)} \tilde{u}_{m-1}^k \cdot n \\ & + \frac{\sqrt{2\pi}}{\text{D}+1} \sum_{j=1}^{m-1} \int_{v \cdot n > 0} \left[L^{\mathcal{D}} \left\{ B_j(U_{m-1-j}^{k,\text{b}}) + K_j(P_0 U_{m-1-j}^{k,\text{int}}) \right\} (v \cdot n) |v|^2 \right]^{\text{tan}} M \, dv \\ & - \frac{1}{(\text{D}+1)\chi} \left\langle (v \cdot n) |v|^2 \left[\sum_{j=4}^m I_j(U_{m-j}^{k,\text{int}}) + \sum_{j=3}^m B_j(U_{m-j}^{k,\text{b}}) \right] \right\rangle^{\text{tan}} + \frac{\text{D}+2}{\text{D}+1} \frac{1}{\chi} \int_0^\infty \langle (\frac{|v|^2}{\text{D}+2} - 1) S_m^{k,\text{bb}} \rangle^{\text{tan}} d\xi. \end{aligned} \quad (8.93)$$

If we express in (8.92) and (8.93) all U^{int} and U^{b} in terms of $U_0^{\text{int}}, P_0 U_1^{\text{int}}, \dots, P_0 U_{m-2}^{\text{int}}$, and collect the corresponding terms, then we can define $V_{m-1}^{\text{u}}(U_0^{k,\text{int}})$ and $V_{m-1}^\theta(U_0^{k,\text{int}})$. Note that $V_1^{\text{u}}, \dots, V_{m-2}^{\text{u}}$ and $V_1^\theta, \dots, V_{m-2}^\theta$ have been defined in the previous rounds of the induction.

Furthermore, the formula (6.8) gives the boundary condition on the normal direction:

$$[u_m^{k,\text{int}} + u_m^{k,\text{b}}] \cdot n = \sum_{j=3}^m V_j^{\text{N}}(P_0 U_{m-j}^{k,\text{int}}), \quad \text{on } \partial\Omega, \quad (8.94)$$

where

$$V_m^N(U_0^{k,\text{int}}) = \int_0^\infty \left\langle \{v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} - i\lambda_0^k\} K_{m-2}(U_0^{k,\text{int}}) \right\rangle d\xi - \sum_{j=1}^{m-3} \int_0^\infty \langle i\lambda_j^k K_{m-2-j}(U_0^{k,\text{int}}) \rangle d\xi.$$

Step 2: Order $O(\sqrt{\varepsilon}^{m-1})$ in the viscous boundary layer.

The equations of $[u_{m-1}^{k,b}]^{\text{tan}}$ and $\theta_{m-1}^{k,b}$ can be derived by considering the order $O(\varepsilon^{m-1})$ of the viscous boundary layer:

$$\mathcal{L}g_{m+1}^{k,b} = v \cdot \nabla_x d\partial_\zeta g_m^{k,b} + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} g_{m-1}^{k,b} - \sum_{j=0}^{m-1} i\lambda_j^k g_{m-1-j}^{k,b}, \quad (8.95)$$

the solvability of which is the following system of ODEs:

$$-\mathcal{A}^d U_m^{k,b} = \sum_{j=0}^{m-1} (\mathcal{F}_j - i\lambda_j^k) U_{m-1-j}^{k,b}, \quad (8.96)$$

where $\mathcal{F}_{m-1}(U_0^{k,b})$ is defined by

$$\mathcal{P} \left\{ v \cdot \nabla_x d\partial_\zeta B_m(U_0^{k,b}) + v \cdot \nabla_x \pi^\alpha \partial_{\pi^\alpha} B_{m-1}(U_0^{k,b}) \right\} = (1, v, \frac{|v|^2}{2} - \frac{D}{2}) \mathcal{F}_{m-1}(U_0^{k,b}). \quad (8.97)$$

Note that the linear operator $\mathcal{F}_1, \dots, \mathcal{F}_{m-2}$ have been defined in the previous rounds of the induction. In particular, $\mathcal{F}_0 = \mathcal{A}^\pi + \mathcal{D}^d$.

Projecting the system (8.96) on $\text{Null}(\mathcal{A}^d)^\perp$, the u-component and ρ -component of which give the equations of $\rho_m^{k,b} + \theta_m^{k,b}$ and $u_m^{k,b} \cdot \nabla_x d$ respectively:

$$-\partial_\zeta (\rho_m^{k,b} + \theta_m^{k,b}) = \left\{ \nu(2 - \frac{2}{D}) \partial_{\zeta\zeta}^2 - i\lambda_0^k \right\} (u_{m-1}^{k,b} \cdot \nabla_x d) + \sum_{j=1}^{m-1} \left\{ (I - \Pi^d)(\mathcal{F}_j - i\lambda_j^k) U_{m-1-j}^{k,b} \right\}^u, \quad (8.98)$$

and

$$\begin{aligned} & -\partial_\zeta (u_m^{k,b} \cdot \nabla_x d) \\ &= \text{div}(u_{m-1}^{k,b} \cdot \nabla_x \pi) + \kappa \partial_{\zeta\zeta}^2 \theta_{m-1}^{k,b} - i\lambda_0^k \frac{D}{D+2} (\rho_{m-1}^{k,b} + \theta_{m-1}^{k,b}) + \sum_{j=1}^{m-1} \left\{ (I - \Pi^d)(\mathcal{F}_j - i\lambda_j^k) U_{m-1-j}^{k,b} \right\}^\rho. \end{aligned} \quad (8.99)$$

Here we use the notation for a vector $V = (V^\rho, V^u, V^\theta)^\top$. Integrating (8.98) from ζ to ∞ gives

$$\rho_m^{k,b} + \theta_m^{k,b} = \sum_{j=2}^m Y_j^b(\zeta, P_0 U_{m-j}^{\text{int}}), \quad (8.100)$$

in which the linear operator Y_2^b, \dots, Y_{m-1}^b have been defined in the previous rounds of the induction. This corresponds to (\mathbf{P}_m^1) .

Next projecting the system (8.96) on $\text{Null}(\mathcal{A}^d)$, the u-component of which gives that $u_{m-1}^{k,b} \cdot \nabla_x \pi$ satisfies the ODE

$$\begin{aligned} & \left\{ \nu \partial_{\zeta\zeta}^2 - i\lambda_0^k \right\} u = -\partial_\pi (\rho_{m-1}^{k,b} + \theta_{m-1}^{k,b}) - \sum_{j=1}^{m-1} \left\{ \Pi^d(\mathcal{F}_j - i\lambda_j^k) U_{m-1-j}^{k,b} \right\}^u, \\ & [u - \frac{\nu}{\chi} \partial_\zeta u](\zeta = 0) = -u_{m-1}^{k,\text{int}} \cdot \nabla_x \pi + \sum_{j=1}^{m-1} V_j^u(P_0 U_{m-1-j}^{\text{int}}), \\ & \lim_{\zeta \rightarrow \infty} u = 0, \end{aligned} \quad (8.101)$$

the solution of which is

$$\mathbf{u}_{m-1}^{k,b} \cdot \nabla_x \pi = \sum_{j=0}^{m-1} \tilde{Z}_j^{b,u}(\zeta, P_0 U_{m-1-j}^{\text{int}}), \quad (8.102)$$

where the linear operator $\tilde{Z}_0^{b,u}, \dots, \tilde{Z}_{m-2}^{b,u}$ have been defined in the previous rounds of the induction. This corresponds to (\mathbf{P}_m^2) .

Projecting the system (8.96) on $\text{Null}(\mathcal{A})$ gives that $\theta_{m-1}^{k,b}$ satisfies the ODE

$$\begin{aligned} \left\{ \kappa \partial_{\zeta}^2 - i \lambda_0^k \right\} \theta &= -i \lambda_0^k \frac{2}{D+2} (\rho_{m-1}^{k,b} + \theta_{m-1}^{k,b}) - \sum_{j=1}^{m-1} \left\{ \Pi^d(\mathcal{F}_j - i \lambda_j^k) U_{m-1-j}^{k,b} \right\}^{\theta}, \\ [\theta - \frac{D+2}{D+1} \frac{\kappa}{\chi} \partial_{\zeta} \theta](\zeta = 0) &= -\theta_{m-1}^{k,\text{int}} + \sum_{j=1}^{m-1} V_j^{\theta}(P_0 U_{m-1-j}^{\text{int}}), \\ \lim_{\zeta \rightarrow \infty} \theta &= 0, \end{aligned} \quad (8.103)$$

the solution of which is

$$\theta_{m-1}^{k,b} = \sum_{j=0}^{m-1} \tilde{Z}_j^{b,\theta}(\zeta, P_0 U_{m-1-j}^{\text{int}}), \quad (8.104)$$

where the linear operator $\tilde{Z}_0^{b,\theta}, \dots, \tilde{Z}_{m-2}^{b,\theta}$ have been defined in the previous rounds of the induction. This corresponds to (\mathbf{P}_m^3) .

Having (8.102) and (8.104), go back to (8.99) and integrate from ζ to ∞ , we obtain

$$\mathbf{u}_m^{k,b} \cdot \nabla_x \mathbf{d} = \sum_{j=1}^m \tilde{Z}_j^b(\zeta, P_0 U_{m-j}^{\text{int}}). \quad (8.105)$$

In particular, by taking $\zeta = 0$ in (8.105), we obtain

$$-\mathbf{u}_m^{k,b} \cdot \mathbf{n} = \sum_{j=1}^m Z_j^b(P_0 U_{m-j}^{\text{int}}), \quad (8.106)$$

Thus from (8.106) and (8.94) we derive the boundary condition of $\mathbf{u}_m^{k,\text{int}} \cdot \mathbf{n}$ which will be used in the next step to solve $U_m^{k,\text{int}}$. This corresponds to (\mathbf{P}_m^4) .

Under these conditions, the equation (8.95) can be solved as $g_{m+1}^{k,b} = \sum_{h=0}^{m+1} B_h(U_{j-h}^{\text{int}})$, where $B_h(U^{\text{int}})$ is defined in (8.80). Note that B_0, B_1, \dots, B_m have been determined in the previous rounds of the induction. This corresponds to (\mathbf{P}_m^5) .

Finally we go back to the kinetic boundary equation (8.86) to solve $g_m^{k,\text{bb}}$ as

$$g_m^{k,\text{bb}} = \sum_{j=1}^m K_j(v, \xi, P_0 U_{m-j}^{\text{int}}), \quad (8.107)$$

where the linear operator $K_m(v, \xi, U_0^{k,\text{int}})$ is the solution to the linear kinetic boundary layer equation (6.3)-(6.4) with the source term $s_m^{\text{bb}}(U_0^{k,\text{int}})$ and the boundary source term $h_m^{\text{bb}}(U_0^{k,\text{int}})$. Note that K_1, \dots, K_{m-1} have been defined in the previous rounds of the induction. This corresponds to (\mathbf{P}_m^6) . Thus we finish **Step 2**.

Step 3: Order $O(\sqrt{\varepsilon}^m)$ in the interior.

The order $O(\varepsilon^{\frac{m}{2}})$ in the interior part of the ansatz yields

$$\mathcal{L} g_{m+2}^{k,\text{int}} = v \cdot \nabla_x g_m^{k,\text{int}} - \sum_{j=0}^m i \lambda_{j,0}^k g_{m-j}^{k,\text{int}}, \quad (8.108)$$

and the solvability condition of which is

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k)U_m^{k,\text{int}} &= \sum_{j=1}^m (i\lambda_j^k - \mathcal{G}_j)U_{m-j}^k, \quad \text{in } \Omega, \\ \mathbf{u}_m^{k,\text{int}} \cdot \mathbf{n} &= \sum_{j=1}^m (Z_j^b + V_j^N)(P_0 U_{m-j}^{k,\text{int}}), \quad \text{on } \partial\Omega, \end{aligned} \quad (8.109)$$

where the vector-valued linear operator \mathcal{G}_j for $j \geq 4$ is defined as

$$(1, v, \frac{|v|^2}{2} - \frac{D}{2})\mathcal{G}_j(U_0^{k,\text{int}}) = \mathcal{P} \left\{ v \cdot \nabla_x I_j(U_0^{k,\text{int}}) \right\}. \quad (8.110)$$

Note that $\mathcal{G}_1 = 0$, $\mathcal{G}_2 = \mathcal{D}$, $\mathcal{G}_3 = 0$ and $V_j^N = 0$ for $j = 1, 2$.

Applying Lemma 5.1 to (8.109), and recalling in (8.82) the definition of Q_j for $j = 1, 2, \dots, m-1$, the formula (5.20) gives

$$\begin{aligned} i\lambda_m^k &= \sqrt{\frac{D+2}{2D}} \int_{\partial\Omega} (Z_m^b + V_m^N)(U_0^{k,\text{int}}) \Psi^k d\sigma_x + \sum_{j=2}^m \left\langle \mathcal{G}_j(Z_{m-j}^{\text{int}}(U_0^{k,\text{int}})) | U_0^{k,\text{int}} \right\rangle \\ &\quad + \sum_h^{m-1} Q_h(P_0 U_{m-h}^{k,\text{int}}, \Psi^k). \end{aligned} \quad (8.111)$$

The orthogonality condition (8.85) implies that the second line of (8.111) vanishes. Thus we can define the righthand side of the first line of (8.111) as $Q_m(\Psi^k, \Psi^k)$. Thus we verify that $i\lambda_m = Q_m(\Psi^k, \Psi^k)$ which is completely determined. This corresponds to (\mathbf{P}_m^7) .

To solve the equation (8.109), we need to consider $m+1$ cases:

Case 1: $i\lambda_h^k$ is a simple eigenvalue of L_h for $0 \leq h \leq m-1$. No orthogonality condition is needed, and every term is fully determined;

Case j ($2 \leq j \leq m+1$): $i\lambda_h^k$ is a multiple eigenvalue of L_h for $0 \leq h \leq j-2$, and a simple eigenvalue of L_h for $j-1 \leq h \leq m-1$.

We only consider the **case m+1** here, i.e. all the eigenvalues $i\lambda_h^k$ are multiple. The other cases are simpler. Taking the inner product with $U_0^{l,\text{int}}$, for $l \neq k$, $\lambda_0^l = \lambda_0^k$, which is

$$\sum_{h=1}^{m-1} i\lambda_h^k a_{m-h}^{kl} = \sum_{h=1}^{m-1} Q_h(P_0 U_{m-h}^{k,\text{int}}, \Psi^l) + Q_m(\Psi^k, \Psi^l). \quad (8.112)$$

If $\Psi^k, \Psi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{m-1}(\lambda_{m-1})$, then because of the orthogonality condition (8.84) for $1 \leq h \leq m-2$,

$$\begin{aligned} Q_h(P_0 U_{m-h}^{k,\text{int}}, \Psi^l) &= Q_h(P_{h-1} U_{m-h}^{k,\text{int}}, \Psi^l) + Q_h\left(\sum_{\delta=1}^{h-1} P_{\delta}^{\perp} U_{m-h}^{k,\text{int}}, \Psi^l\right) \\ &= i\lambda_h^l a_{m-h}^{kl}. \end{aligned}$$

For $h = m-1$, $Q_{m-1}(P_0 U_1^{k,\text{int}}, \Psi^l) = i\lambda_{m-2}^l a_1^{kl} + Q_{m-1}(P_{m-2}^{\perp} U_1^{k,\text{int}}, \Psi^l)$. Thus, the identity (8.112) implies that we need the orthogonality condition that for $k \neq l$,

$$Q_m(\Psi^k, \Psi^l) = \int_{\Omega} L_m(\Psi^k) \Psi^l dx = 0, \quad \text{for } \Psi^k, \Psi^l \in H_1 \cap \dots \cap H_{m-1}, \quad (8.113)$$

where the symmetric operator L_m is defined by $L_m \Psi^l = i\lambda_m^l \Psi^l$, for $\Psi^l \in H_1 \cap \dots \cap H_{m-1}$.

If $\Psi^k, \Psi^l \in H_1(\lambda_1) \cap H_2(\lambda_2) \cap \dots \cap H_{m-2}(\lambda_{m-2}) \cap H_{m-1}^{\perp}(\lambda_{m-1})$, i.e. $\lambda_h^k = \lambda_h^l$ for $0 \leq h \leq m-2$, but $\lambda_{m-1}^k \neq \lambda_{m-1}^l$, from the identity (8.112), for these k, l , a_1^{kl} can be determined by

$$a_1^{kl} = \frac{1}{i\lambda_{m-1}^k - i\lambda_{m-1}^l} Q_m(\Psi^k, \Psi^l).$$

This means that $(P_0^\perp + P_1^\perp + \dots + P_{m-1}^\perp)U_1^{k,\text{int}}$ is completely determined, but $P_{m-1}U_1^{k,\text{int}}$ is still left as undetermined.

If $\Psi^k, \Psi^l \in H_1 \cap \dots \cap H_{m-3} \cap H_{m-2}^\perp$,

$$Q_{m-1}(P_{m-1}^\perp U_1^{k,\text{int}}, \Psi^l) + Q_m(\Psi^k, \Psi^l) = (i\lambda_{m-2}^k - i\lambda_{m-2}^l)a_2^{kl} + i\lambda_{m-1}^k a_1^{kl}, \quad (8.114)$$

from which a_2^{kl} thus $P_{m-2}^\perp U_2^{k,\text{int}}$ is completely determined.

Under these solvability conditions, the equation (8.109) can be solved as

$$U_m^{k,\text{int}} = P_0 U_m^{k,\text{int}} + \sum_{h=1}^m Z_h^{\text{int}}(P_0 U_{m-h}^{k,\text{int}}),$$

where $Z_m^{\text{int}}(U_0^{k,\text{int}})$ is the solution to the following equation:

$$\begin{aligned} (\mathcal{A} - i\lambda_0^k)U &= \sum_{h=1}^m (i\lambda_h^k - \mathcal{G}_h)Z_{m-h}^{\text{int}}(U_0^{k,\text{int}}), \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= (Z_m^{\text{b}} + V_m^{\text{N}})U_0^{k,\text{int}}, \quad \text{on } \partial\Omega. \end{aligned} \quad (8.115)$$

Thus, $U_m^{k,\text{int}}$ is determined modulo $P_0 U_m^{k,\text{int}}, P_1 U_{m-1}^{k,\text{int}}, \dots, P_{m-1} U_1^{k,\text{int}}$ which are undetermined at this stage. Under these conditions, the equation (8.108) is solved as $g_{m+2}^{k,\text{int}} = I_0(U_{m+2}^{\text{int}}) + I_2(U_m^{\text{int}}) + \sum_{h=4}^{m+2} I_{m+2-h}(U_{m+2-h}^{\text{int}})$. This corresponds to (\mathbf{P}_m^8) , (\mathbf{P}_m^9) and (\mathbf{P}_m^{10}) .

We can now inductively continue the process, namely go to the order $O(\sqrt{\varepsilon}^{m-1})$ of the kinetic boundary layer, the order $O(\sqrt{\varepsilon}^m)$ of the viscous boundary layer, then the order $O(\sqrt{\varepsilon}^{m+1})$ of the interior, and so on. We should do this at least till the order $N+2$ where N is the precision of the error in (7.10). Note however, that for a given $\lambda = \lambda_0^k$, we may only need to construct a small number of the L_j if after few steps all the eigenvalues become simple, namely if for some j all the eigenvalues of L_j are simple on the space $H_1(\lambda_1) \cap \dots \cap H_{j-1}(\lambda_{j-1})$. It is clear that if the eigenvalues become simple for some $j \leq N+2$, then the orthogonality condition (8.84) allows to determine the eigenfunctions Ψ^k uniquely. If the process does not end, then we just need to satisfy the condition till the order $N+2$ which yield a non-unique choice of eigenfunctions. Also, in this case, we set all the undetermined pieces of the eigenfunction, namely those left undetermined to be zero.

9. PROOF OF PROPOSITION 7.1: TRUNCATION ERROR ESTIMATES

In the previous sections, we construct the kinetic-fluid boundary layers up to any order for $\alpha_\varepsilon = \sqrt{2\pi}\chi\varepsilon^\beta$. Now we define the approximated eigenfunction and eigenvalues $g_{\varepsilon,N}^k$ and $\lambda_{\varepsilon,N}^k$ by truncation in the corresponding ansatz. More specifically,

$$\begin{aligned} g_{\varepsilon,N}^k &= \sum_{j=0}^N \left\{ g_j^{k,\text{int}} + g_j^{k,\text{b}} \right\} \varepsilon^{\frac{j}{2}} + \sum_{j=1}^N g_j^{k,\text{bb}} \varepsilon^{\frac{j}{2}}, \\ \lambda_{\varepsilon,N}^k &= \sum_{j=0}^N \lambda_j^k \sqrt{\varepsilon}^j. \end{aligned}$$

9.1. Estimates of $R_{\varepsilon,N}^k$. Using the eigen-equation (7.4), we can easily find that the error term $R_{\varepsilon,N}^k$ has the form of

$$\begin{aligned} R_{\varepsilon,N}^k &= \{ (i\lambda_0^k - v \cdot \nabla_x) g_{N-1}^{k,\text{int}} + (i\lambda_0^k - v \cdot \nabla_x \pi) \partial_\pi [g_{N-1}^{k,\text{b}} + g_{N-1}^{k,\text{bb}}] - v \cdot \nabla_x d \partial_\zeta g_N^{k,\text{b}} \\ &\quad + \sum_{j=1}^{N-1} i\lambda_j^k \hat{g}_{N-1-j}^k \} \varepsilon^{\frac{N-1}{2}} + \text{“higher order terms”}, \end{aligned}$$

where $\hat{g}^k = g^{k,\text{int}} + g^{k,\text{b}} + g^{k,\text{bb}}$. From the constructions of $g^{k,\text{int}}$, $g^{k,\text{b}}$ and $g^{k,\text{bb}}$, it is easy to know that

$$\|g_j^k\|_{L^r(dx; L^p(aM dv))} \leq C,$$

for all j , and $1 < r, p < \infty$, where g^k stands for $g^{k,\text{int}}$, $g^{k,\text{b}}$ or $g^{k,\text{bb}}$.

Indeed, both the hydrodynamic and the kinetic parts of $g^{k,\text{int}}$ and $g^{k,\text{b}}$ have coefficients in terms of the components of $U^{k,\text{int}}$ and $U^{k,\text{b}}$. From Lemma 5.1, the solutions $U_j^{k,\text{int}}$ of the equation (5.16) can be represented linearly in terms of components of U_i^{int} for $0 \leq i < j$ and the boundary terms of $U_i^{k,\text{b}}$ and $g_i^{k,\text{b}}$ for $0 \leq i < j$. Note that the pseudo inverse operator $(\mathcal{A} - i\lambda^{\tau,k})^{-1}$ is bounded, and furthermore, the boundary values of $U_i^{k,\text{b}}$ and $g_i^{k,\text{b}}$ are linearly in terms of $U_l^{k,\text{int}}$ for $0 \leq l \leq i$. For $U_j^{k,\text{b}}$, their components are solutions of second order ordinary differential equations with boundary conditions in terms of $U_i^{k,\text{int}}$ and $g_i^{k,\text{b}}$ for $0 \leq i \leq j$. Moreover, the solutions of the linear kinetic boundary layer equation (6.3) for $g_i^{k,\text{bb}}$ are bounded in $L^r(dx, L^p(aM dv))$ in terms of $U_l^{k,\text{int}}$ and $U_l^{k,\text{b}}$ for $0 \leq l < i$. So all $g^{k,\text{int}}$, $g^{k,\text{b}}$ or $g^{k,\text{bb}}$ are linearly depends on components of $U_0^{k,\text{int}}$, i.e. Ψ^k and $\nabla_x \Psi^k$ which is the eigenfunctions of $-\Delta_x$ with Neumann boundary conditions. From the basic regularity theory of elliptic operator, they are bounded in $L^r(dx; \Omega)$ for any $1 < r \leq \infty$.

For the term $(v \cdot \nabla_x d) \partial_\zeta g_N^{k,\text{bb}}$, we integrate over $\Omega \times \mathbb{R}^D$ and use simple change of variable $(y_1, y_2, \dots, y_{D-1}) = \pi(x), y_D = \frac{d(x)}{\sqrt{\varepsilon}}$, we can have extra $\sqrt{\varepsilon}$, so all will be in the higher order terms. Thus, we have the error estimate (7.10).

9.2. Estimates of $g_{\varepsilon,N}^k - g_0^{k,\text{int}}$. The leading order term of $g_{\varepsilon,N}^k - g_0^{k,\text{int}}$ is $g_0^{k,\text{b}}$, so using the expressions above for $u_0^{k,\text{b}}$, $\theta_0^{k,\text{b}}$ and a simple change of variable which will give an extra $\sqrt{\varepsilon}$, we have

$$\|g_{\varepsilon,N}^k - g_0^{k,\text{int}}\|_{L^r(dx, L^p(aM dv))} \leq C\varepsilon^{\frac{1}{2r}}.$$

Thus we get (7.11).

9.3. Boundary error estimate. Finally, the boundary error term $r_{\varepsilon,N}^k$ is

$$r_{\varepsilon,N}^k = -\sqrt{\varepsilon}^{N+1} L^{\mathcal{D}}(g_N^{k,\text{int}} + g_N^{k,\text{b}} + g_N^{k,\text{bb}})$$

from which we can get the estimate (7.12). Thus we finish the proof of the Proposition 7.1.

10. PROOF OF THE WEAK CONVERGENCE IN THEOREM 3.1 AND 3.2

In order to derive the fluid equation with the boundary conditions, we need to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (2.3). We choose the renormalization used in [27]:

$$\Gamma(Z) = \frac{Z - 1}{1 + (Z - 1)^2}. \quad (10.1)$$

After multiplying $\Gamma'(G_\varepsilon)$ and dividing by ε , equation (2.19) becomes

$$\partial_t \tilde{g}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g}_\varepsilon = \frac{1}{\varepsilon} \Gamma'(G_\varepsilon) \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} q_\varepsilon b(\omega, v_1 - v) d\omega M_1 dv_1, \quad (10.2)$$

where $\tilde{g}_\varepsilon = \frac{1}{\varepsilon} \Gamma(G_\varepsilon)$ can be considered as the L^2 part of the fluctuations g_ε and q_ε is the scaled collision integrand defined as

$$q_\varepsilon = \frac{G'_{\varepsilon 1} G'_\varepsilon - G_{\varepsilon 1} G_\varepsilon}{\varepsilon^2}. \quad (10.3)$$

By introducing $N_\varepsilon = 1 + \varepsilon^2 g_\varepsilon^2$, we can write

$$\tilde{g}_\varepsilon = \frac{g_\varepsilon}{N_\varepsilon}, \quad \Gamma'(G_\varepsilon) = \frac{2}{N_\varepsilon^2} - \frac{1}{N_\varepsilon}.$$

When moments of the renormalized Boltzmann equation (10.2) are formally taken with respect to any $\zeta \in \text{span}\{1, v_1, \dots, v_D, |v|^2\}$, one obtains the local conservation laws with defects

$$\begin{aligned} \partial_t \tilde{\rho}_\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot \tilde{\mathbf{u}}_\varepsilon &= \frac{1}{\varepsilon} \langle \Gamma'(G_\varepsilon) q_\varepsilon \rangle, \\ \partial_t \tilde{\mathbf{u}}_\varepsilon + \frac{1}{\varepsilon} \nabla_x (\tilde{\rho}_\varepsilon + \tilde{\theta}_\varepsilon) + \frac{1}{\varepsilon} \nabla_x \cdot \langle \mathbf{A}(v) \tilde{g}_\varepsilon \rangle &= \frac{1}{\varepsilon} \langle v \Gamma'(G_\varepsilon) q_\varepsilon \rangle, \\ \partial_t \tilde{\theta}_\varepsilon + \frac{1}{\varepsilon} \frac{2}{D} \nabla_x \cdot \tilde{\mathbf{u}}_\varepsilon + \frac{2}{D} \frac{1}{\varepsilon} \nabla_x \cdot \langle \mathbf{B}(v) \tilde{g}_\varepsilon \rangle &= \frac{1}{\varepsilon} \left\langle \left(\frac{|v|^2}{D} - 1 \right) \Gamma'(G_\varepsilon) q_\varepsilon \right\rangle, \end{aligned} \quad (10.4)$$

which can be written as

$$\partial_t \tilde{U}_\varepsilon + \frac{1}{\varepsilon} \mathcal{A} \tilde{U}_\varepsilon + \tilde{Q}_\varepsilon = \tilde{R}_\varepsilon, \quad (10.5)$$

where

$$\begin{aligned} \tilde{U}_\varepsilon &= (\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\theta}_\varepsilon) = (\langle \tilde{g}_\varepsilon \rangle, \langle v \tilde{g}_\varepsilon \rangle, \langle (\frac{|v|^2}{D} - 1) \tilde{g}_\varepsilon \rangle), \\ \tilde{Q}_\varepsilon &= (0, \frac{1}{\varepsilon} \nabla_x \cdot \langle \mathbf{A}(v) \tilde{g}_\varepsilon \rangle, \frac{1}{\varepsilon} \nabla_x \cdot \langle \mathbf{B}(v) \tilde{g}_\varepsilon \rangle), \end{aligned}$$

and the local conservation defect

$$\tilde{R}_\varepsilon = \frac{1}{\varepsilon} \left\langle (1, v, \frac{|v|^2}{D} - 1) \Gamma'(G_\varepsilon) q_\varepsilon \right\rangle.$$

Notice that we do not know if $\tilde{\mathbf{u}}_\varepsilon \cdot \mathbf{n} = 0$, so \tilde{U}_ε is not necessary in the domain of \mathcal{A} for every $\varepsilon > 0$, thus the notation $\mathcal{A} \tilde{U}_\varepsilon$ in (10.5) is not quite rigorous. However we can show that the weak limit of $\tilde{\mathbf{u}}_\varepsilon$, say, \mathbf{u} , satisfies $\mathbf{u} \cdot \mathbf{n} = 0$ on the boundary, also see [23]. From the local conservation laws with defect (10.5), formally the limit of \tilde{U}_ε will be in the null space of the acoustic operator \mathcal{A} . In other words, any weak limits of $(\tilde{\rho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\theta}_\varepsilon)$ will satisfy the incompressibility $\nabla_x \cdot \tilde{\mathbf{u}} = 0$ and Boussinesq relation $\tilde{\rho} + \tilde{\theta} = 0$.

The term $\frac{1}{\varepsilon} \mathcal{A} \tilde{U}_\varepsilon$ in (10.5) describes the acoustic waves with propagation speed $\frac{1}{\varepsilon}$. As ε goes to zero, the sound waves propagate faster and faster to make the fluid limit singular. To derive the incompressible fluid equations, a natural way is to project the local conservation laws (10.5) onto $\text{Null}(\mathcal{A})$ and $\text{Null}(\mathcal{A})^\perp$ respectively. First \tilde{U}_ε can be orthogonally decomposed as

$$\begin{aligned} \tilde{U}_\varepsilon &= \Pi \tilde{U}_\varepsilon + \Pi^\perp \tilde{U}_\varepsilon \\ &= \left(\langle (1 - \frac{|v|^2}{D+2}) \tilde{g}_\varepsilon \rangle, \mathbb{P} \langle v \tilde{g}_\varepsilon \rangle, \langle (\frac{|v|^2}{D+2} - 1) \tilde{g}_\varepsilon \rangle \right) \\ &\quad + \left(\langle \frac{|v|^2}{D+2} \tilde{g}_\varepsilon \rangle, \mathbb{Q} \langle v \tilde{g}_\varepsilon \rangle, \langle \frac{2|v|^2}{D(D+2)} \tilde{g}_\varepsilon \rangle \right), \end{aligned} \quad (10.6)$$

in which we call $\Pi \tilde{U}_\varepsilon$ and $\Pi^\perp \tilde{U}_\varepsilon$ the incompressible and acoustic parts of \tilde{U}_ε respectively.

By definition of Leray projection in a bounded domain (5.1), the boundary conditions of $\mathbb{P} \langle v \tilde{g}_\varepsilon \rangle$ and $\mathbb{Q} \langle v \tilde{g}_\varepsilon \rangle$ are

$$\mathbb{P} \langle v \tilde{g}_\varepsilon \rangle \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbb{Q} \langle v \tilde{g}_\varepsilon \rangle \cdot \mathbf{n} = \tilde{\mathbf{u}}_\varepsilon \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega.$$

To derive the weak form of the evolution equations of $\Pi \tilde{U}_\varepsilon$, we take the test function Y in (2.24) as special infinitesimal Maxwellian in the incompressible mode:

$$Y^{incom}(x, v) = -\chi + w \cdot v + \chi \left(\frac{|v|^2}{2} - \frac{D}{2} \right),$$

where $(\chi, w) \in C^\infty(\overline{\Omega}, \mathbb{R}^D \times \mathbb{R})$ with $\nabla_x \cdot w = 0$ in Ω and $w \cdot n = 0$ on $\partial\Omega$. Because χ and w are independent, the weak form of (10.5) can be written separately as:

$$\begin{aligned} & \int_{\Omega} \mathbb{P}\langle v\tilde{g}_\epsilon(t_2) \rangle \cdot w \, dx - \int_{\Omega} \mathbb{P}\langle v\tilde{g}_\epsilon(t_1) \rangle \cdot w \, dx \\ & - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle A\tilde{g}_\epsilon \rangle : \nabla_x w \, dx \, dt + \frac{1}{\sqrt{2\pi\varepsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \gamma\tilde{g}_\epsilon(w \cdot v) \rangle_{\partial\Omega} \, d\sigma_x \, dt \\ & = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} w \cdot \langle \langle v\Gamma'(G_\epsilon)q_\epsilon \rangle \rangle \, dx \, dt, \end{aligned} \quad (10.7)$$

and

$$\begin{aligned} & \frac{D+2}{2} \int_{\Omega} \left\langle \left(\frac{|v|^2}{D+2} - 1 \right) \tilde{g}_\epsilon(t_2) \right\rangle \chi \, dx - \frac{D+2}{2} \int_{\Omega} \left\langle \left(\frac{|v|^2}{D+2} - 1 \right) \tilde{g}_\epsilon(t_1) \right\rangle \chi \, dx \\ & - \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle B\tilde{g}_\epsilon \rangle \cdot \nabla_x \chi \, dx \, dt + \frac{1}{\sqrt{2\pi\varepsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left\langle \chi \left(\frac{|v|^2}{D+2} - 1 \right) \gamma\tilde{g}_\epsilon \right\rangle_{\partial\Omega} \, d\sigma_x \, dt \\ & = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \chi \left\langle \left(\frac{|v|^2}{D+2} - 1 \right) \Gamma'(G_\epsilon)q_\epsilon \right\rangle \, dx \, dt. \end{aligned} \quad (10.8)$$

Identities (10.7) and (10.8) are the local conservation laws in the incompressible modes. It is the starting point of the proof of the weak convergence to the incompressible Navier-Stokes equations with boundary conditions in the Main Theorems 3.1 and 3.2. It has been proved in [27] the convergence of the interior terms of (10.7) and (10.8) as $\varepsilon \rightarrow 0$ to recover the weak form of incompressible Navier-Stokes equations. It is only left to derive the boundary conditions of the limiting equations.

The strategy to recover the boundary conditions in the limit is basically the same as [33] except some necessary modifications. For the convenience of the readers, and also because we work in more general collision kernels, we briefly go through the proof here. In [27], the author proved that inside the domain $\mathbb{R}^+ \times \Omega \times \mathbb{R}^D$, the family of fluctuations g_ε is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(\sigma M dv dx))$, and that every limit point g has the form (3.4). Lemma 5.1 of [33] showed that the trace of the limit point γg belongs to $L^1_{loc}(dt; L^1(M|v \cdot n(x)| d\sigma_x))$ and satisfies

$$\gamma g = v \cdot \gamma u + \left(\frac{|v|^2}{2} - \frac{D+2}{2} \right) \gamma \theta, \quad (10.9)$$

where γu and $\gamma \theta$ denote the fluid traces of u and θ .

We list some key *a priori* estimates from [33] on γg_ε . The first one is from the inside, we generalize it to the more general collision kernel case considered in this paper.

Lemma 10.1. *For all $p > 0$, as $\varepsilon \rightarrow 0$,*

$$\gamma \tilde{g}_\epsilon \rightarrow \gamma g \quad \text{in } w\text{-}L^1_{loc}(dt; w\text{-}L^1(M(1 + |v|^p)|v \cdot n(x)| dv d\sigma_x)). \quad (10.10)$$

Proof. The proof is essentially the same as Lemma 5.2 of [33], except for some new argument to treat the soft potential collision kernel case. First, using the function

$$\Gamma(Z) = \left(\frac{Z - 1}{1 + (Z - 1)^2} \right)^{5/3}$$

in the renormalized formulation (2.22) gives

$$(\varepsilon \partial_t + v \cdot \nabla_x) \tilde{g}_\epsilon^{5/3} = \frac{5}{3} \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} \tilde{g}_\epsilon^{2/3} q_\epsilon \left(\frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon} \right) b(\omega, v - v_1) d\omega M_1 dv_1. \quad (10.11)$$

To estimate the right-hand side in (10.11), we apply the classical Young's inequality, namely,

$$pz \leq r^*(p) + r(z),$$

for every p and z in the domains of r^* and r . Here the function r is defined over $z > -1$ by $r(z) = z \log(1+z)$ which is strictly convex, and r^* is the Legendre dual of r .

$$\begin{aligned} \left| \frac{q_\varepsilon}{N_\varepsilon^2} |\tilde{g}_\varepsilon|^{2/3} \right| &\leq \frac{1}{\varepsilon^4} G_\varepsilon G_{\varepsilon 1} r \left(\frac{\varepsilon^2 q_\varepsilon}{G_\varepsilon G_{\varepsilon 1}} \right) + \frac{1}{\varepsilon^4} G_\varepsilon G_{\varepsilon 1} r^* \left(\frac{\varepsilon^2 \tilde{g}_\varepsilon^{2/3}}{N_\varepsilon^2} \right) \\ &\leq \frac{1}{\varepsilon^4} G_\varepsilon G_{\varepsilon 1} r \left(\frac{\varepsilon^2 q_\varepsilon}{G_\varepsilon G_{\varepsilon 1}} \right) + G_\varepsilon G_{\varepsilon 1} \frac{|\tilde{g}_\varepsilon|^{4/3}}{N_\varepsilon^4} r^*(1), \end{aligned} \quad (10.12)$$

The second inequality above used the superquadratic homogeneity of r^* . By the entropy dissipation rate bound, the first term on the right-hand side of (10.12) is bounded in $L^1_{loc}(dt, L^1(d\nu dx))$. Since $N_\varepsilon \geq 1$ and $G_\varepsilon \leq \sqrt{2N_\varepsilon}$, the integral of the second term can be bounded as follows:

$$\begin{aligned} &\sqrt{2} \iint_{\mathbb{R}^D \times \mathbb{R}^D} |\tilde{g}_\varepsilon|^{4/3} G_{\varepsilon 1} \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv \\ &\leq \sqrt{2} \iint_{\mathbb{R}^D \times \mathbb{R}^D} \frac{|\tilde{g}_\varepsilon|^{4/3}}{N_\varepsilon^3 \sqrt{N_\varepsilon}} (1 + \varepsilon |\tilde{g}_{\varepsilon 1}|) \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv \\ &+ \sqrt{2} \iint_{\mathbb{R}^D \times \mathbb{R}^D} |\tilde{g}_\varepsilon|^{4/3} \varepsilon |g_{\varepsilon 1} - \tilde{g}_{\varepsilon 1}| \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} a_1 M_1 dv_1 a M dv. \end{aligned} \quad (10.13)$$

Using the *assumption 3*, namely (2.7) and $|\varepsilon \tilde{g}_\varepsilon| \leq \frac{1}{2}$, the first term in (10.13) is bounded. Indeed, it is bounded by

$$C \int_{\mathbb{R}^D} \frac{|\tilde{g}_\varepsilon|^{4/3}}{N_\varepsilon^3 \sqrt{N_\varepsilon}} a M dv \leq C \int_{\mathbb{R}^D} \frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}} a M dv \leq C.$$

The second term in (10.13) is bounded as

$$\begin{aligned} &\sqrt{2} \iint_{\mathbb{R}^D \times \mathbb{R}^D} |\varepsilon \tilde{g}_\varepsilon|^{4/3} \varepsilon^{2/3} \frac{|g_{\varepsilon 1}|}{\sqrt{N_{\varepsilon 1}}} \frac{g_{\varepsilon 1}^2}{\sqrt{N_{\varepsilon 1}}} \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} a M dv a_1 M_1 dv_1 \\ &\leq \varepsilon^{2/3} 2 \left(\frac{1}{2} \right)^{4/3} \int_{\mathbb{R}^D} \frac{g_{\varepsilon 1}^2}{\sqrt{N_{\varepsilon 1}}} a_1 M_1 dv_1. \end{aligned} \quad (10.14)$$

Since the righthand side of (10.14) vanishes as ε goes to zero, we deduce that $(\varepsilon \partial_t + v \cdot \nabla_x) \tilde{g}_\varepsilon^{5/3}$ is uniformly bounded in $L^1_{loc}(dt, L^1(M dv dx))$. The rest of the proof of (10.10) is the same as that of Lemma 5.2 in [33]. \square

The next lemma is the *a priori* estimate of γg_ε from the boundary term in the entropy inequality (2.27). The proof is the same as Lemma 6.1 in [33] with some trivial modification. So we just state the lemma without giving the proof.

Lemma 10.2. *Define $\gamma_\varepsilon = \gamma_+ g_\varepsilon - \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}$ and*

$$\gamma_\varepsilon^{(1)} = \gamma_\varepsilon \mathbf{1}_{\gamma_+ G_\varepsilon \leq 2 \langle G_\varepsilon \rangle_{\partial\Omega} \leq 4 \gamma_+ G_\varepsilon}, \quad \gamma_\varepsilon^{(2)} = \gamma_\varepsilon - \gamma_\varepsilon^{(1)}. \quad (10.15)$$

Then each of these is bounded as follows:

$$\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma_\varepsilon^{(1)}}{[1 + \varepsilon^2 (\gamma_+ g_\varepsilon)^2]^{\frac{1}{4}}} \quad \text{in} \quad L^2_{loc}(dt; L^2(M |v \cdot n(x)| dv d\sigma_x)), \quad (10.16)$$

$$\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma_\varepsilon^{(1)}}{[1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2]^{\frac{1}{4}}} \quad \text{in} \quad L_{loc}^2(dt; L^2(M|v \cdot n(x)|dv d\sigma_x)), \quad (10.17)$$

$$\frac{\alpha_\varepsilon}{\varepsilon^2} \gamma_\varepsilon^{(2)} \quad \text{in} \quad L_{loc}^1(dt; L^1(M|v \cdot n(x)|dv d\sigma_x)). \quad (10.18)$$

Using Lemma 10.1 and Lemma 10.2, we can prove the following lemma which describes how to define the renormalized outgoing mass flux $\mathbf{1}_{\Sigma_+} \rho$.

Lemma 10.3. *Assume that $\frac{\alpha_\varepsilon}{\sqrt{2\pi\varepsilon}} \rightarrow \chi \in (0, +\infty]$. Then up to the extraction of a subsequence,*

$$\frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \quad \text{and} \quad \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2}$$

converge in $w\text{-}L_{loc}^1(dt; w\text{-}L^1(M|v \cdot n(x)|dv dx))$ and have the same weak limit. Moreover, there exists $\rho \in L_{loc}^1(dt; L^1(d\sigma_x))$ such that, up to the extraction of a subsequence,

$$\frac{\mathbf{1}_{\Sigma_+} \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \rightarrow \mathbf{1}_{\Sigma_+} \rho \quad \text{in} \quad w\text{-}L_{loc}^1(dt; w\text{-}L^1(M|v \cdot n(x)|dv dx)).$$

Furthermore,

$$\rho = \langle \gamma_+ g \rangle_{\partial\Omega}.$$

Lemma 10.3 is nothing but Lemma 6.2 and Lemma 6.3 in [33]. The proof is basically the same except some trivial modifications because we use some different renormalizations. Thus we skip the proof here.

Now it is ready to recover the Dirichlet boundary condition. For the case $\frac{\alpha_\varepsilon}{\varepsilon} \rightarrow \infty$, from (10.16) and (10.18), we deduce that

$$\begin{aligned} \frac{\gamma_\varepsilon^{(1)}}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2} &\rightarrow 0 \quad \text{strongly in} \quad L_{loc}^2(dt; L^2(M|v \cdot n(x)|dv dx)), \\ \frac{\gamma_\varepsilon^{(2)}}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2} &\rightarrow 0 \quad \text{strongly in} \quad L_{loc}^1(dt; L^1(M|v \cdot n(x)|dv dx)); \end{aligned}$$

hence, we get

$$\frac{\gamma_\varepsilon}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2} \rightarrow 0 \quad \text{strongly in} \quad L_{loc}^1(dt; L^1(M|v \cdot n(x)|dv dx)). \quad (10.19)$$

On the other hand,

$$\frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} = \gamma_+ \tilde{g}_\varepsilon - \frac{\mathbf{1}_{\Sigma_+} \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \rightarrow \gamma_+ g - \mathbf{1}_{\Sigma_+} \rho, \quad (10.20)$$

in $w\text{-}L_{loc}^1(dt; w\text{-}L^1(M|v \cdot n(x)|dv dx))$. Then (10.19) and (10.20) imply that

$$\gamma_+ g = \mathbf{1}_{\Sigma_+} \rho$$

where ρ depends only on (t, x) . Thus, by (10.9) we get the Dirichlet boundary condition

$$\gamma u = 0 \quad \text{and} \quad \gamma \theta = 0.$$

Now, we concentrate on the Navier boundary condition case. Using the previous convergence results, we can take limits in the conservation laws (10.7) and (10.8) to get the weak form of the boundary conditions. In the weak forms (10.7) and (10.8), the limits of the interior terms have been carried in [27], which is stated in the following lemma:

Lemma 10.4. Assume that $\frac{\alpha_\epsilon}{\sqrt{2\pi\epsilon}} \rightarrow \chi \in [0, \infty)$, then up to the extraction of a sequence, $\mathbb{P}\langle v\tilde{g}_\epsilon \rangle$ and $\langle (\frac{|v|^2}{D+2} - 1)\tilde{g}_\epsilon \rangle$ converge to u and θ in $C([0, \infty); w-L^1(dx))$ such that, for all $w \in C^\infty(\bar{\Omega}; \mathbb{R}^D)$ with $\nabla_x \cdot w = 0$ in Ω and $w \cdot n = 0$ on $\partial\Omega$, for all $\chi \in C^\infty(\bar{\Omega}; \mathbb{R})$, and for all $t_1, t_2 > 0$,

$$\begin{aligned} & \int_{\Omega} u(t_2) \cdot w \, dx - \int_{\Omega} u(t_1) \cdot w \, dx - \int_{t_1}^{t_2} \int_{\Omega} \sum_{i,j} u_i u_j \partial_i w_j \, dx dt \\ & + \nu \int_{t_1}^{t_2} \int_{\Omega} \sum_{i,j} (\partial_i u_j + \partial_j u_i) \partial_i w_j \, dx dt \end{aligned} \quad (10.21)$$

$$\begin{aligned} & = - \lim_{\epsilon \rightarrow 0} \frac{\alpha_\epsilon}{\sqrt{2\pi\epsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left\langle \frac{\gamma_\epsilon^{(1)}(w \cdot v) \mathbf{1}_{|v|^2 \leq 20|\log \epsilon|}}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} \right\rangle_{\partial\Omega} d\sigma_x dt, \\ & \int_{\Omega} \theta(t_2) \cdot \chi \, dx - \int_{\Omega} \theta(t_1) \cdot \chi \, dx - \int_{t_1}^{t_2} \int_{\Omega} \theta u \cdot \nabla_x \chi \, dx dt + \frac{2}{D+2} \kappa \int_{t_1}^{t_2} \int_{\Omega} \nabla_x \theta \cdot \nabla_x \chi \, dx dt \\ & = - \lim_{\epsilon \rightarrow 0} \frac{\alpha_\epsilon}{\sqrt{2\pi\epsilon}} \int_{t_1}^{t_2} \int_{\partial\Omega} \left\langle \frac{\gamma_\epsilon^{(1)}}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} \chi \left(\frac{|v|^2}{D+2} - 1 \right) \mathbf{1}_{|v|^2 \leq 20|\log \epsilon|} \right\rangle_{\partial\Omega} d\sigma_x dt. \end{aligned} \quad (10.22)$$

where $\gamma_+ \hat{g}_\epsilon = (1 - \alpha_\epsilon) \gamma_+ g_\epsilon + \alpha_\epsilon \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega}$.

Proof. It is the analogue of Lemma 7.1 and Lemma 7.2 of [33] and the idea of the proof is the same. Denote by Y_ϵ the test function $(w \cdot v) \mathbf{1}_{|v|^2 \leq 20|\log \epsilon|}$ or $\chi(\frac{|v|^2}{D+2} - 1) \mathbf{1}_{|v|^2 \leq 20|\log \epsilon|}$. Then Y_ϵ has the property: $Y_\epsilon = LY_\epsilon$, recalling L is the local reflection operator defined in (2.2). From (2.25), the renormalized form of the Maxwell boundary condition reads

$$\gamma_- \tilde{g}_\epsilon = (1 - \alpha_\epsilon) \frac{L\gamma_+ g_\epsilon}{1 + \epsilon^2 (L\gamma_+ \hat{g}_\epsilon)^2} + \alpha_\epsilon \frac{\langle \gamma_+ g_\epsilon \rangle_{\partial\Omega}}{1 + \epsilon^2 (L\gamma_+ \hat{g}_\epsilon)^2}, \quad (10.23)$$

where

$$\begin{aligned} \gamma_+ \hat{g}_\epsilon &= (1 - \alpha_\epsilon) \gamma_+ g_\epsilon + \alpha_\epsilon \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega}, \\ &= \gamma_+ g_\epsilon - \alpha_\epsilon \gamma_\epsilon. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\epsilon} \int_{\partial\Omega} \langle \gamma \tilde{g}_\epsilon Y_\epsilon \rangle_{\partial\Omega} d\sigma_x &= \frac{1}{\epsilon} \int_{\partial\Omega} \left\langle \frac{\epsilon^2 \gamma_+ g_\epsilon (\gamma_+ \hat{g}_\epsilon^2 - \gamma_+ g_\epsilon^2)}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} Y_\epsilon \mathbf{1}_{\Sigma_+} \right\rangle_{\partial\Omega} d\sigma_x \\ &+ \frac{\alpha_\epsilon}{\epsilon} \int_{\partial\Omega} \left\langle \frac{\gamma_\epsilon}{1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2} Y_\epsilon \mathbf{1}_{\Sigma_+} \right\rangle_{\partial\Omega} d\sigma_x \\ &= \frac{\alpha_\epsilon}{\epsilon} \int_{\partial\Omega} \left\langle \frac{\gamma_\epsilon^{(1)} + \gamma_\epsilon^{(2)}}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} Y_\epsilon \mathbf{1}_{\Sigma_+} \right\rangle_{\partial\Omega} d\sigma_x \\ &- \frac{\alpha_\epsilon}{\epsilon} \int_{\partial\Omega} \left\langle \frac{(\gamma_\epsilon^{(1)} + \gamma_\epsilon^{(2)}) \epsilon^2 (\gamma_+ g_\epsilon \gamma_+ \hat{g}_\epsilon)}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} Y_\epsilon \mathbf{1}_{\Sigma_+} \right\rangle_{\partial\Omega} d\sigma_x. \end{aligned} \quad (10.24)$$

By (10.18),

$$\begin{aligned} & \int_{t_1}^{t_2} \left| \frac{\alpha_\epsilon}{\epsilon} \int_{\partial\Omega} \left\langle \frac{\gamma_\epsilon^{(2)}}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} Y_\epsilon \mathbf{1}_{\Sigma_+} \right\rangle_{\partial\Omega} d\sigma_x \right| dt \\ & \leq C\epsilon \left\| \frac{Y_\epsilon}{(1 + \epsilon^2 \gamma_+ g_\epsilon^2)(1 + \epsilon^2 \gamma_+ \hat{g}_\epsilon^2)} \right\|_\infty \leq C\epsilon |\log \epsilon|. \end{aligned} \quad (10.25)$$

The $\gamma_\varepsilon^{(2)}$ part in the last term of (10.24) can be estimated as (10.25). For the $\gamma_\varepsilon^{(1)}$ part, from (10.16),

$$\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma_\varepsilon^{(1)}}{\sqrt{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2}} \text{ is relatively compact in } w\text{-}L_{loc}^1(dt; w\text{-}L^1(M|v \cdot n| dv d\sigma_x)) \quad (10.26)$$

Use the fact that

$$\sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\varepsilon \gamma_+ g_\varepsilon \varepsilon \hat{g}_\varepsilon}{\sqrt{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2 (1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)}} \quad (10.27)$$

is bounded in L^∞ and goes to 0 a.e. Then by the Product Limit Theorem of [3], the product of (10.26) and (10.27) goes to 0 in $L_{loc}^1(dt)$ as $\varepsilon \rightarrow 0$. Thus we finish the proof of the lemma. \square

Now, it is ready to recover the Navier boundary condition by taking limit in the last terms in (10.21) and (10.22). As in [33], we can deduce that

$$\begin{aligned} \frac{\alpha_\varepsilon}{\sqrt{2\pi\varepsilon}} \left\langle \frac{\gamma_\varepsilon^{(1)}(w \cdot v) \mathbf{1}_{|v|^2 \leq 20|\log \varepsilon|}}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)} \right\rangle_{\partial\Omega} &\rightarrow \lambda \langle (\gamma_+ g - \mathbf{1}_{\Sigma_+} \langle \gamma_+ g \rangle_{\partial\Omega}) (w \cdot v) \rangle_{\partial\Omega}, \\ \frac{\alpha_\varepsilon}{\sqrt{2\pi\varepsilon}} \left\langle \frac{\gamma_\varepsilon^{(1)}}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)} \chi \left(\frac{|v|^2}{D+2} - 1 \right) \mathbf{1}_{|v|^2 \leq 20|\log \varepsilon|} \right\rangle_{\partial\Omega} \\ &\rightarrow \lambda \langle (\gamma_+ g - \mathbf{1}_{\Sigma_+} \langle \gamma_+ g \rangle_{\partial\Omega}) \chi \left(\frac{|v|^2}{D+2} - 1 \right) \rangle_{\partial\Omega} \end{aligned}$$

in $w\text{-}L_{loc}^1(dt; w\text{-}L^1(d\sigma_x))$. Use (10.9), we finally prove the weak form of the incompressible Navier-Stokes equations with Navier boundary conditions:

$$\begin{aligned} &\int_\Omega u(t_2) \cdot w \, dx - \int_\Omega u(t_1) \cdot w \, dx - \int_{t_1}^{t_2} \int_\Omega \sum_{i,j} u_i u_j \partial_i w_j \, dx dt \\ &+ \nu \int_{t_1}^{t_2} \int_\Omega \sum_{i,j} (\partial_i u_j + \partial_j u_i) \partial_i w_j \, dx dt \\ &= \lambda \int_{t_1}^{t_2} \int_{\partial\Omega} \gamma u \cdot w \, d\sigma_x dt, \\ &\int_\Omega \theta(t_2) \cdot \chi \, dx - \int_\Omega \theta(t_1) \cdot \chi \, dx - \int_{t_1}^{t_2} \int_\Omega \theta u \cdot \nabla_x \chi \, dx dt \\ &+ \frac{2}{D+2} \kappa \int_{t_1}^{t_2} \int_\Omega \nabla_x \theta \cdot \nabla_x \chi \, dx dt = \frac{D+1}{D+2} \alpha \int_{t_1}^{t_2} \int_{\partial\Omega} \gamma \theta \chi \sigma_x \, dt. \end{aligned}$$

Thus we finish the proof of the weak convergence results in the Theorem 3.1 and Theorem 3.2.

11. PROOF OF THE STRONG CONVERGENCE IN THEOREM 3.1

In the previous section, we proved that the incompressible part of the fluid moments \tilde{U}_ε , i.e. $\Pi \tilde{U}_\varepsilon$ converges only *weakly* to solutions of the incompressible NSF equations. This weak convergence is caused by the persistence of fast acoustic part $\Pi^\perp \tilde{U}_\varepsilon$, as in the periodic domain [27]. If $\Pi^\perp \tilde{U}_\varepsilon$ vanishes in some strong sense as ε goes to zero, we can improve the convergence of $\Pi \tilde{U}_\varepsilon$ from weak to strong. The main novelty of this paper is to prove that in the bounded domain Ω , when $\alpha_\varepsilon = O(\sqrt{\varepsilon})$, the acoustic part will be damped *instantaneously*. This damping effect comes from the kinetic-fluid coupled boundary layers. More precisely, we have the following proposition:

Proposition 11.1. *Let $\Pi^\perp \tilde{U}_\varepsilon$ be defined as (10.6). If $\alpha_\varepsilon = O(\sqrt{\varepsilon})$, then*

$$\Pi^\perp \tilde{U}_\varepsilon \rightarrow 0 \quad \text{in } L^2_{loc}(dt; L^2(dx)),$$

as $\varepsilon \rightarrow 0$.

This proposition is also true for $\alpha_\varepsilon = O(\varepsilon^\beta)$, $0 \leq \beta < \frac{1}{2}$ and $\frac{1}{2} < \beta < 1$. These cases will be treated in a separate paper.

Now we apply Proposition 11.1 to prove the Main Theorem 3.1, and leave its proof to the next subsection.

11.1. Strong Convergence in L^1 : Proof of Theorem 3.1. We first show that we can improve the relative compactness of the family of fluctuations g_ε from weak to strong in $L^1_{loc}(dt; L^1(\sigma M dv dx))$. Indeed, g_ε can be decomposed as

$$\begin{aligned} g_\varepsilon &= \mathcal{P} \tilde{g}_\varepsilon + \mathcal{P}^\perp \tilde{g}_\varepsilon + \frac{\varepsilon^2 g_\varepsilon^3}{N_\varepsilon} \\ &= v \cdot \mathbb{P} \tilde{u}_\varepsilon + \left(\frac{D}{D+2} \tilde{\theta}_\varepsilon - \frac{2}{D+2} \tilde{\rho}_\varepsilon \right) \left(\frac{|v|^2}{2} - \frac{D+2}{2} \right) + v \cdot \mathbb{Q} \tilde{u}_\varepsilon + \frac{|v|^2}{D+2} \left(\tilde{\rho}_\varepsilon + \tilde{\theta}_\varepsilon \right) \\ &\quad + \mathcal{P}^\perp \tilde{g}_\varepsilon + \frac{\varepsilon^2 g_\varepsilon}{\sqrt{N_\varepsilon}} \frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}}, \end{aligned}$$

where \mathcal{P} is the projection to $\text{Null}(\mathcal{L})$ defined in (2.11), \mathbb{P} is the Leray projection, and $\mathbb{Q} = \mathbb{I} - \mathbb{P}$.

It has been proved in [27] that $\mathcal{P}^\perp \tilde{g}_\varepsilon \rightarrow 0$ in $L^2_{loc}(dt; L^2(aM dv dx))$, (see (6.41) in [27]). We can also show that

$$\mathbb{P} \tilde{u}_\varepsilon \rightarrow u, \quad \frac{D}{D+2} \tilde{\theta}_\varepsilon - \frac{2}{D+2} \tilde{\rho}_\varepsilon \rightarrow \theta, \quad \text{in } L^2_{loc}(dt; L^2(dx)). \quad (11.1)$$

Indeed, this convergence is justified in Lemma 5.6 in [19]. Although the renormalization and decomposition of g_ε are different in [19] and the current paper, the proof of the convergence (11.1) can follow the argument in the proof of Lemma 5.6 in [19]. Furthermore, the Proposition 11.1 yields that

$$v \cdot \mathbb{P}^\perp \tilde{u}_\varepsilon + \frac{|v|^2}{D+2} \left(\tilde{\rho}_\varepsilon + \tilde{\theta}_\varepsilon \right) \rightarrow 0 \quad \text{in } L^2_{loc}(dt; L^2(M dv dx)).$$

Thus $\mathcal{P} \tilde{g}_\varepsilon \rightarrow g = v \cdot u + \left(\frac{1}{2} |v|^2 - \frac{D+2}{2} \right) \theta$ in $L^2_{loc}(dt; L^2(M dv dx))$, as $\varepsilon \rightarrow 0$. The key nonlinear estimate in [3] claims that

$$\sigma \frac{g_\varepsilon^2}{\sqrt{N_\varepsilon}} = O(|\log \varepsilon|) \quad \text{in } L^\infty(dt; L^1(aM dv dx)).$$

It is easy to see that $\frac{\varepsilon \tilde{g}_\varepsilon}{\sqrt{N_\varepsilon}}$ is bounded, hence

$$\frac{\varepsilon^2 g_\varepsilon^3}{N_\varepsilon} \rightarrow 0 \quad \text{in } L^1_{loc}(dt; L^1(\sigma M dv dx)). \quad (11.2)$$

We deduce that g_ε is relatively compact in $L^1_{loc}(dt; L^1(\sigma M dv dx))$ and that every limit g has the form (3.4), combining the above estimates.

Next, we can also improve the convergence of the moments of g_ε . In [27], it was proved that the incompressible part $(\mathbb{P} \langle v g_\varepsilon \rangle, \langle (\frac{1}{D+2} |v|^2 - 1) g_\varepsilon \rangle)$ converge to (u, θ) in $C([0, \infty); w-L^1(dx))$. We also have $(\mathbb{P} \langle v g_\varepsilon \rangle, \langle (\frac{1}{D+2} |v|^2 - 1) g_\varepsilon \rangle)$ converge to (u, θ) in $L^2_{loc}(dt; L^2(dx))$. Now, from Proposition 11.1, we know that the acoustic part $\mathbb{Q} \langle v \tilde{g}_\varepsilon \rangle$ and $\langle (\frac{1}{D+2} |v|^2 \tilde{g}_\varepsilon) \rangle$ converge strongly to 0 in $L^2_{loc}(dt; L^2(dx))$. So combining this with (11.2), we get

$$\begin{aligned} \langle v g_\varepsilon \rangle &\rightarrow u \quad \text{in } L^1_{loc}(dt; L^1(dx; \mathbb{R}^D)) \cap C([0, \infty); w-L^1(dx; \mathbb{R}^D)), \\ \langle (\frac{1}{D} |v|^2 - 1) g_\varepsilon \rangle &\rightarrow \theta \quad \text{in } L^1_{loc}(dt; L^1(dx; \mathbb{R})) \cap C([0, \infty); w-L^1(dx; \mathbb{R})). \end{aligned}$$

Furthermore, since now we have $\tilde{u}_\varepsilon \rightarrow u$ and $\tilde{\theta} \rightarrow \theta$ in $L^2_{loc}(dt; L^2(dx))$, we can improve the Quadratic Limit Theorem 13.1 in [27] to

$$\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon \rightarrow u \otimes u, \quad \tilde{\theta}_\varepsilon \tilde{u}_\varepsilon \rightarrow u\theta, \quad \tilde{\theta}_\varepsilon^2 \rightarrow \theta^2 \quad \text{in } L^1_{loc}(dt; L^1(dx)), \quad (11.3)$$

as $\varepsilon \rightarrow 0$.

Let $s \in (0, \infty]$ be from the assumed bound (2.7) on b . Let $p = 2 + \frac{1}{s-1}$, so that $p = 2$ when $s = \infty$. Let $\hat{\xi} \in L^p(aMdv)$ be such that $\mathcal{P}\hat{\xi} = 0$ and set $\xi = \mathcal{L}\hat{\xi}$, hence,

$$\frac{1}{\varepsilon} \langle \xi \tilde{g}_\varepsilon \rangle = \frac{1}{\varepsilon} \langle \xi \mathcal{P}^\perp \tilde{g}_\varepsilon \rangle = \langle \hat{\xi} \mathcal{Q}(\tilde{g}_\varepsilon, \tilde{g}_\varepsilon) \rangle - \langle \langle \hat{\xi} \tilde{g}_\varepsilon \rangle \rangle + \langle \langle \hat{\xi} T_\varepsilon \rangle \rangle.$$

We know from in [27] that

$$\langle \langle \hat{\xi} T_\varepsilon \rangle \rangle \rightarrow 0 \quad \text{in } L^1_{loc}(dt; L^1(dx)), \quad (11.4)$$

and

$$\langle \langle \hat{\xi} \tilde{g}_\varepsilon \rangle \rangle \rightarrow \langle \hat{\xi} \hat{A} : \nabla_x u + \langle \hat{\xi} \hat{B} \rangle \cdot \nabla_x \theta \rangle \quad \text{in } w\text{-}L^2_{loc}(dt; w\text{-}L^2(dx)). \quad (11.5)$$

Note that

$$\begin{aligned} \langle \hat{\xi} \mathcal{Q}(\tilde{g}_\varepsilon, \tilde{g}_\varepsilon) \rangle &= \langle \hat{\xi} \mathcal{Q}(\mathcal{P}\tilde{g}_\varepsilon, \mathcal{P}\tilde{g}_\varepsilon) \rangle + 2\langle \hat{\xi} \mathcal{Q}(\mathcal{P}\tilde{g}_\varepsilon, \mathcal{P}^\perp \tilde{g}_\varepsilon) \rangle \\ &\quad + \langle \hat{\xi} \mathcal{Q}(\mathcal{P}^\perp \tilde{g}_\varepsilon, \mathcal{P}\tilde{g}_\varepsilon) \rangle. \end{aligned}$$

It is easy to show that the last two terms above vanish as $\varepsilon \rightarrow 0$. For the first term,

$$\begin{aligned} \langle \hat{\xi} \mathcal{Q}(\mathcal{P}\tilde{g}_\varepsilon, \mathcal{P}\tilde{g}_\varepsilon) \rangle &= \frac{1}{2} \langle \xi \mathcal{P}^\perp (\mathcal{P}\tilde{g}_\varepsilon)^2 \rangle \\ &= \frac{1}{2} \langle \xi \hat{A} : (\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon) + \langle \xi \hat{B} \rangle \cdot \tilde{u}_\varepsilon \tilde{\theta}_\varepsilon + \frac{1}{2} \langle \xi \hat{C} \rangle \tilde{\theta}_\varepsilon^2 \rangle. \end{aligned} \quad (11.6)$$

Applying the quadratic limit (11.3), (11.6) can be taken limit in $L^1_{loc}(dt; L^1(dx))$ strongly. Combining with convergence (11.4) and (11.5), we get

$$\frac{1}{\varepsilon} \langle \xi \mathcal{P}^\perp \tilde{g}_\varepsilon \rangle \rightarrow \left\langle \xi \left(\frac{1}{2} \hat{A} : u \otimes u + \hat{B} \cdot u\theta + \frac{1}{2} \hat{C} \theta^2 - \hat{A} : \nabla_x u - \hat{B} \cdot \nabla_x \theta \right) \right\rangle$$

in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(dx))$. Since $g_\varepsilon - \tilde{g}_\varepsilon \rightarrow 0$ in $L^\infty(dt; L^1(\sigma M dv dx))$, the convergence above implies (3.8). Thus we finish the proof of the Main Theorem 3.1.

11.2. Proof of Proposition 11.1. We will reduce the proof of the Proposition 11.1 to show that the projection of \tilde{U}_ε on each *fixed* acoustic mode goes to zero in $L^2_{loc}(dt; L^2(dx))$. We know that $\Pi^\perp \tilde{U}_\varepsilon$ is uniformly bounded in $L^\infty(dt; L^2(dx))$, so it can be represented as

$$\begin{aligned} \Pi^\perp \tilde{U}_\varepsilon &= \sum_{k \in \mathbb{N}} \langle \tilde{U}_\varepsilon, U^{+,k} \rangle_{\mathbb{H}} U^{+,k} + \langle \tilde{U}_\varepsilon, U^{-,k} \rangle_{\mathbb{H}} U^{-,k} \\ &= \frac{D+2}{2D} \sum_{k \in \mathbb{N}} \left(\frac{2D}{(D+2)^2} \int_{\Omega} \langle |v|^2 \tilde{g}_\varepsilon \rangle \overline{\Psi^k} dx \Psi^k \right. \\ &\quad \left. + \frac{2}{\int_{\Omega} \langle v \tilde{g}_\varepsilon \rangle \cdot \frac{\nabla_x \overline{\Psi^k}}{i\lambda^k} dx} \frac{\nabla_x \Psi^k}{i\lambda^k} \right), \end{aligned} \quad (11.7)$$

recalling that the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ is defined in (5.2), $U^{+,k}$ and $U^{-,k}$ are defined in (5.7).

The above summation includes infinitely many terms. To reduce the problem to a finite number of modes, we need some regularity in x of $\langle v \tilde{g}_\varepsilon \rangle$ and $\langle |v|^2 \tilde{g}_\varepsilon \rangle$. The tool adapted to investigating this property is the velocity averaging theorem given in [14] and the improvement to L^1 averaging in [18].

Following the similar argument in the proof of Proposition 11.2 in [27], and apply to (10.2), we can show that for each $\zeta \in \text{Span}\{1, v, |v|^2\}$ and $T > 0$, there exists a function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{z \rightarrow 0^+} \eta(z) = 0$,

$$\| \langle \zeta \tilde{g}_\varepsilon(t, x + y, v) - \zeta \tilde{g}_\varepsilon(t, x, v) \rangle \|_{L^2([0, T] \times \Omega)} \leq \eta(|y|), \quad (11.8)$$

for every $y \in \Omega$ such that $|y| \leq 1$, uniformly in $\varepsilon \in [0, 1]$. From the classical criterion of compactness in L^2 , $\langle v \tilde{g}_\varepsilon \rangle$ and $|v|^2 \tilde{g}_\varepsilon$ are relatively compact in $L^2_{loc}(dt, L^2(dx))$ which implies that

$$\sum_{k > N} \int_0^T \left| \langle \tilde{U}_\varepsilon, U^{\tau, k} \rangle_{\mathbb{H}} \right|^2 dt \leq C_N \|\Pi^\perp \tilde{U}_\varepsilon\|_{L^2([t_1, t_2]; L^2(dx))} \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (11.9)$$

recalling from (11.8) that $C_N \rightarrow 0$ as $N \rightarrow \infty$. (11.9) implies that, to show $\Pi^\perp \tilde{U}_\varepsilon \rightarrow 0$ strongly in $L^2_{loc}(dt, L^2(dx))$, we need only to prove that $\langle \tilde{U}_\varepsilon, U^{\tau, k} \rangle_{\mathbb{H}}$ converges strongly to 0 in $L^2(0, T)$ for any fixed acoustic mode k . Furthermore, the relation

$$\langle \tilde{U}_\varepsilon, U^{\tau, k} \rangle_{\mathbb{H}} = \int_{\Omega} \left\langle \tilde{g}_\varepsilon, g_0^{\tau, k, \text{int}} \right\rangle dx$$

implies that the proof of Proposition 11.1 is reduced to showing that :

Proposition 11.2. *Assume that $\alpha_\varepsilon = O(\sqrt{\varepsilon})$ and let \tilde{g}_ε be the renormalized fluctuation defined in (10.3), satisfying the scaled Boltzmann equation (10.2), and $g_0^{\tau, k, \text{int}}$ (τ is + or -) be the infinitesimal Maxwellian of acoustic mode $k \geq 1$:*

$$g_0^{\tau, k, \text{int}} = \frac{D}{D+2} \Psi^k + \frac{\nabla_x \Psi^k}{\tau i \lambda^k} \cdot v + \frac{2}{D+2} \Psi^k \left(\frac{|v|^2}{2} - \frac{D}{2} \right).$$

Then, for any fixed mode k ,

$$\int_{\Omega} \left\langle \tilde{g}_\varepsilon, g_0^{\tau, k, \text{int}} \right\rangle dx \rightarrow 0 \quad \text{in } L^2(0, T), \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We start from the weak formulation of the rescaled Boltzmann equation (2.24) with the renormalization Γ defined in (10.1) and the test function Y taken to be the approximate eigenfunctions of \mathcal{L}_ε constructed in Proposition 7.1 to the order $N = 4$, namely $Y = g_{\varepsilon, 4}^{\tau, k}$:

$$\begin{aligned} & \int_{\Omega} \langle \tilde{g}_\varepsilon(t_2) g_{\varepsilon, 4}^{\tau, k} \rangle dx - \int_{\Omega} \langle \tilde{g}_\varepsilon(t_1) g_{\varepsilon, 4}^{\tau, k} \rangle dx \\ & + \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle \tilde{g}_\varepsilon \mathcal{L}_\varepsilon g_{\varepsilon, 4}^{\tau, k} \rangle dx dt + \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \gamma \tilde{g}_\varepsilon \gamma g_{\varepsilon, 4}^{\tau, k} (v \cdot n) \rangle d\sigma_x dt \\ & = \frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\Omega} \langle \langle R_\varepsilon g_{\varepsilon, 4}^{\tau, k} \rangle \rangle dx dt, \end{aligned} \quad (11.10)$$

where

$$R_\varepsilon = \Gamma'(G_\varepsilon) q_\varepsilon + \frac{1}{\varepsilon} \left(\frac{g_{\varepsilon 1}}{N_{\varepsilon 1}} + \frac{g_\varepsilon}{N_\varepsilon} - \frac{g'_{\varepsilon 1}}{N'_{\varepsilon 1}} - \frac{g'_\varepsilon}{N'_\varepsilon} \right).$$

Define

$$\tilde{b}_\varepsilon^{\tau, k}(t) = \int_{\Omega} \langle \tilde{g}_\varepsilon(t) g_{\varepsilon, 4}^{\tau, k} \rangle dx.$$

Then from (11.10) $\tilde{b}_\varepsilon^{\tau, k}(t)$ satisfies

$$\tilde{b}_\varepsilon^{\tau, k}(t_2) - \tilde{b}_\varepsilon^{\tau, k}(t_1) - \frac{1}{\varepsilon} \overline{i \lambda_{\varepsilon, 4}^{\tau, k}} \int_{t_1}^{t_2} \tilde{b}_\varepsilon^{\tau, k}(t) dt = \int_{t_1}^{t_2} c_\varepsilon^{\tau, k}(t) dt, \quad (11.11)$$

where $c_\varepsilon^{\tau, k}(t)$ is:

$$\begin{aligned} c_\varepsilon^{\tau, k}(t) &= -\frac{1}{\varepsilon} \int_{\Omega} \langle \tilde{g}_\varepsilon(t) R_{\varepsilon, 4}^{\tau, k} \rangle dx - \frac{1}{\varepsilon} \int_{\partial\Omega} \langle \gamma \tilde{g}_\varepsilon \gamma g_{\varepsilon, 4}^{\tau, k} (v \cdot n) \rangle d\sigma_x \\ &+ \frac{1}{\varepsilon} \int_{\Omega} \langle \langle R_\varepsilon g_{\varepsilon, 4}^{\tau, k} \rangle \rangle dx. \end{aligned} \quad (11.12)$$

We claim that the boundary contribution in (11.12) is zero as $\varepsilon \rightarrow 0$, i.e.

Lemma 11.1. *Let $g_{\varepsilon,4}^{\tau,k}$ be the approximate eigenfunction of \mathcal{L}_ε constructed in Proposition 7.1. Then,*

$$\frac{1}{\varepsilon} \int_{\partial\Omega} \langle \gamma \tilde{g}_\varepsilon \gamma g_{\varepsilon,4}^{\tau,k}(v \cdot n) \rangle d\sigma_x = \Gamma_1^{\tau,k} + \Gamma_2^{\tau,k}, \quad (11.13)$$

where $\Gamma_1^{\tau,k}$ is bounded in $L_{loc}^p(dt)$ for $p > 1$, and $\Gamma_2^{\tau,k}$ vanishes in $L_{loc}^1(dt)$ as $\varepsilon \rightarrow 0$.

We leave the proof of Lemma 11.1 to the section 11.5.

11.3. Estimates of $c_\varepsilon^{\tau,k}$. We will decompose $c_\varepsilon^{\tau,k}(t)$ into two parts: one is vanishing in $L_{loc}^1(dt)$, the other is bounded in $L_{loc}^p(dt)$ for some $p > 1$. First, taking $N = 4$, $r = p = 2$ in (7.10) and noticing the $L_{loc}^\infty(dt, L^2(aM dv dx))$ boundedness of \tilde{g}_ε , we have the estimate of the first term in (11.12):

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_{\Omega} \langle \tilde{g}_\varepsilon(t) R_{\varepsilon,4}^{\tau,k} \rangle dx \right| &\leq \frac{1}{\varepsilon} \|R_{\varepsilon,4}^{\tau,k}\|_{L^2(\frac{1}{a} M dv dx)} \|\tilde{g}_\varepsilon\|_{L^2(aM dv dx)} \\ &\leq C \sqrt{\varepsilon}. \end{aligned}$$

Second, Lemma 11.1 implies that one part of the boundary term in (11.12), namely $\Gamma_2^{\tau,k}$ in (11.13) will be vanishing in $L_{loc}^1(dt)$ as ε goes to zero. The third term in (11.12) is estimated as follows:

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{\Omega} \langle R_\varepsilon g_{\varepsilon,4}^{\tau,k} \rangle dx \\ &= \frac{1}{\varepsilon} \int_{\Omega} \langle R_\varepsilon \mathcal{P} g_{\varepsilon,4}^{\tau,k} \rangle dx + \frac{1}{\varepsilon} \int_{\Omega} \langle R_\varepsilon \mathcal{P}^\perp g_{\varepsilon,4}^{\tau,k} \rangle dx. \end{aligned} \quad (11.14)$$

For the first term in the right-hand side of (11.14), because $\mathcal{P} g_{\varepsilon,4}^{\tau,k}$ is in $\text{Null}(\mathcal{L})$, it has the form of

$$\frac{1}{\varepsilon} \int_{\Omega} \langle \Gamma'(G_\varepsilon) q_\varepsilon \zeta \rangle dx + \frac{1}{\varepsilon^2} \int_{\Omega} \langle \mathcal{L} \tilde{g}_\varepsilon \zeta \rangle dx, \quad \text{for some } \zeta(t, x) \in \text{Null}(\mathcal{L}). \quad (11.15)$$

The second term above is zero, the first term converges to zero strongly in $L_{loc}^1(dt)$ as $\varepsilon \rightarrow 0$ by the Conservation Defect Theorem (Proposition 8.1) in [27].

For the second term in the right-hand side of (11.14), from the calculations in Proposition 7.1 again, we have

$$\begin{aligned} \frac{1}{\varepsilon} \mathcal{P}^\perp g_{\varepsilon,4}^{\tau,k} &= \sqrt{\frac{D+2}{2D}} \left(\frac{\nabla_x^2 \Psi^k}{\tau i \lambda^k} : \hat{A} + \frac{2 \nabla_x \Psi^k}{D+2} \cdot \hat{B} \right) + \frac{1}{\sqrt{\varepsilon}} \left(\nabla_x d \otimes \partial_\zeta u_0^{\tau,k,b} : \hat{A} + \partial_\zeta \theta_0^{\tau,k,b} \nabla_x d \cdot \hat{B} \right) \\ &\quad + \frac{1}{\sqrt{\varepsilon}} g_1^{\tau,k,bb} + \text{higher order terms}. \end{aligned} \quad (11.16)$$

We decompose R_ε into

$$R_\varepsilon = T_\varepsilon + (\tilde{g}'_{\varepsilon 1} \tilde{g}'_\varepsilon - \tilde{g}_{\varepsilon 1} \tilde{g}_\varepsilon) + q_\varepsilon \left(\frac{2}{N_\varepsilon^2} - \frac{1}{N_\varepsilon} - \frac{1}{N'_{\varepsilon 1} N'_\varepsilon N_{\varepsilon 1} N_\varepsilon} \right), \quad (11.17)$$

where T_ε is

$$T_\varepsilon = \frac{q_\varepsilon}{N'_{\varepsilon 1} N'_\varepsilon N_{\varepsilon 1} N_\varepsilon} - \frac{1}{\varepsilon} (\tilde{g}'_{\varepsilon 1} + \tilde{g}'_\varepsilon - \tilde{g}_{\varepsilon 1} - \tilde{g}_\varepsilon) - (\tilde{g}'_{\varepsilon 1} \tilde{g}'_\varepsilon - \tilde{g}_{\varepsilon 1} \tilde{g}_\varepsilon).$$

When we integrate (11.16) over Ω , for the second term of (11.16) which is a function of $(\pi(x), \frac{d(x)}{\sqrt{\varepsilon}})$, we make the change of variables:

$$y_1 = \pi^1(x), \dots, y_{D-1} = \pi^{D-1}(x), y_D = \frac{d(x)}{\sqrt{\varepsilon}}. \quad (11.18)$$

Then $dx = \sqrt{\varepsilon} T^* dy$, where

$$T^* = \det^{-1} \begin{pmatrix} \nabla_x \pi \\ \nabla_x d \end{pmatrix} > 0.$$

This extra $\sqrt{\varepsilon}$ cancels with the $\sqrt{\varepsilon}^{-1}$ in the second term of (11.16). Similarly, for the third term of (11.16), we make the change of variables $y_1 = \pi(x)^1, \dots, y_{D-1} = \pi(x)^{D-1}, y_D = \frac{d(x)}{\varepsilon}$, and consequently $dx = \varepsilon T^* dy$. Thus, for the integral of (11.16), the first two terms are the same order, namely $O(1)$, while the third term is of order $O(\sqrt{\varepsilon})$, and the rest term is even higher order in ε . By the Flux Remainder Theorem (Proposition 10.1) of [27], we have

$$\frac{1}{\varepsilon} \int_{\Omega} \langle T_{\varepsilon} \mathcal{P}^{\perp} g_{\varepsilon,4}^{\tau,k} \rangle dx \rightarrow 0, \quad \text{in } L_{loc}^1(dt),$$

as $\varepsilon \rightarrow 0$. Furthermore, from the Bilinear estimates (Lemma 9.1) of [27], we have

$$\frac{1}{\varepsilon} \int_{\Omega} \langle (\tilde{g}'_{\varepsilon 1} \tilde{g}'_{\varepsilon} - \tilde{g}_{\varepsilon 1} \tilde{g}_{\varepsilon 1}) \mathcal{P}^{\perp} g_{\varepsilon,4}^{\tau,k} \rangle dx \leq C \int_{\Omega} \langle a \tilde{g}_{\varepsilon}^2 \rangle dx \leq C.$$

The third term in (11.17) can be written as $q_{\varepsilon}/\sqrt{N_{\varepsilon}}$ times a bounded sequence that vanishes almost everywhere as $\varepsilon \rightarrow 0$. Then the $L^2(d\nu dx dt)$ boundedness of $q_{\varepsilon}/\sqrt{N_{\varepsilon}}$ implies that it times (11.16) is relatively compact in $w\text{-}L_{loc}^1(dt; w\text{-}L^1(dx))$, then following from the Product Limit Theorem of [3],

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\Omega} \langle q_{\varepsilon} \left(\frac{2}{N_{\varepsilon}^2} - \frac{1}{N_{\varepsilon}} - \frac{1}{N'_{\varepsilon 1} N'_{\varepsilon} N_{\varepsilon 1} N_{\varepsilon}} \right) \mathcal{P}^{\perp} g_{\varepsilon,4}^{\tau,k} \rangle dx \\ & \rightarrow 0, \quad \text{in } L_{loc}^1(dt), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Now we can decompose $c_{\varepsilon}^{\tau,k}(t)$ into

$$c_{1,\varepsilon}^{\tau,k}(t) = -\frac{1}{\varepsilon} \int_{\Omega} \langle (\tilde{g}'_{\varepsilon 1} \tilde{g}'_{\varepsilon} - \tilde{g}_{\varepsilon 1} \tilde{g}_{\varepsilon 1}) \mathcal{P}^{\perp} g_{\varepsilon,4}^{\tau,k} \rangle dx - \Gamma_1^{\tau,k},$$

and $c_{2,\varepsilon}^{\tau,k}(t) = c_{\varepsilon}^{\tau,k}(t) - c_{1,\varepsilon}^{\tau,k}(t)$, where $\Gamma_1^{\tau,k}$ appears in (11.13).

The above arguments show that $c_{2,\varepsilon}^{\tau,k}(t) \rightarrow 0$, in $L_{loc}^1(dt)$, and Lemma 11.1 gives that $c_{1,\varepsilon}^{\tau,k}(t)$ is bounded in $L_{loc}^p(dt)$ for some $p > 1$.

11.4. Estimates of $\tilde{b}_{\varepsilon}^{\tau,k}$. From (11.11), $\tilde{b}_{\varepsilon}^{\tau,k}$ satisfies the ordinary differential equation

$$\frac{d}{dt} \tilde{b}_{\varepsilon}^{\tau,k} - \frac{1}{\varepsilon} \overline{i\lambda_{\varepsilon,4}^{\tau,k}} \tilde{b}_{\varepsilon}^{\tau,k} = c_{1,\varepsilon}^{\tau,k}(t) + c_{2,\varepsilon}^{\tau,k}(t). \quad (11.19)$$

The solution to (11.19) is given by

$$\tilde{b}_{\varepsilon}^{\tau,k}(t) = \tilde{b}_{\varepsilon}^{\tau,k}(0) e^{\frac{1}{\varepsilon} \overline{i\lambda_{\varepsilon,4}^{\tau,k}} t} + \int_0^t [c_{1,\varepsilon}^{\tau,k}(s) + c_{2,\varepsilon}^{\tau,k}(s)] e^{-\frac{1}{\varepsilon} \overline{i\lambda_{\varepsilon,4}^{\tau,k}} (s-t)} ds. \quad (11.20)$$

From the Proposition 7.1, $i\lambda_{\varepsilon,4}^{\tau,k} = \tau i\lambda^k + i\lambda_1^{\tau,k} \sqrt{\varepsilon} + i\tilde{\lambda}_1^{\tau,k} \varepsilon$, where $\tilde{\lambda}_1^{\tau,k} = O(1)$.

$$\begin{aligned} \frac{1}{\varepsilon} \overline{i\lambda_{\varepsilon,4}^{\tau,k}} t &= \frac{1}{\sqrt{\varepsilon}} \left[\text{Re}(i\lambda_1^{\tau,k}) + \sqrt{\varepsilon} \text{Re}(i\tilde{\lambda}_1^{\tau,k}) \right] t \\ &\quad - i \left[\tau \frac{1}{\varepsilon} \lambda^k + \frac{1}{\sqrt{\varepsilon}} \text{Im}(i\lambda_1^{\tau,k}) + \text{Im}(i\tilde{\lambda}_1^{\tau,k}) \right] t. \end{aligned} \quad (11.21)$$

Using (11.21), the first term in (11.20) is estimated as follows:

$$\begin{aligned} & \|\tilde{b}_{\varepsilon}^{\tau,k}(0) e^{\frac{1}{\varepsilon} \overline{i\lambda_{\varepsilon,4}^{\tau,k}} t}\|_{L^2(0,T)} \\ &= |\tilde{b}_{\varepsilon}^{\tau,k}(0)| \left[-2 \left(\text{Re}(i\lambda_1^{\tau,k}) + \sqrt{\varepsilon} \text{Re}(i\tilde{\lambda}_1^{\tau,k}) \right) \right]^{-\frac{1}{2}} \left(1 - e^{\frac{1}{\sqrt{\varepsilon}} [\text{Re}(i\lambda_1^{\tau,k}) + \sqrt{\varepsilon} \text{Re}(i\tilde{\lambda}_1^{\tau,k})] T} \right)^{\frac{1}{2}} \varepsilon^{\frac{1}{4}}. \end{aligned}$$

To estimate $|\tilde{b}_{\varepsilon}^{\tau,k}(0)|$, from

$$\tilde{b}_{\varepsilon}^{\tau,k}(0) = \int_{\Omega} \langle \tilde{g}_{\varepsilon}^{\text{in}}, g_0^{k,\text{int}} \rangle dx + \int_{\Omega} \langle \tilde{g}_{\varepsilon}^{\text{in}}, g_{\varepsilon,4}^{\tau,k} - g_0^{k,\text{int}} \rangle dx,$$

noticing that $g_0^{\tau,k,\text{int}} \in \text{Null}(\mathcal{L})$ and $\|\langle \zeta(v) \tilde{g}_\epsilon^{\text{int}} \rangle\|_{L^2(dx)}$ is bounded for every $\zeta(v) \in \text{Null}(\mathcal{L})$, and the error estimate for $g_{\epsilon,4}^{\tau,k} - g_0^{\tau,k,\text{int}}$ in (7.11), we deduce that $|\tilde{b}_\epsilon^{\tau,k}(0)|$ is bounded. Using the key fact that $\text{Re}(i\lambda_1^{\tau,k}) < 0$, we deduce that for any $0 < T < \infty$, sufficiently small ϵ :

$$\|\tilde{b}_\epsilon^{\tau,k}(0)e^{-\frac{1}{\epsilon^2}\overline{i\lambda_{\epsilon,4}^{\tau,k}}t}\|_{L^2(0,T)} \leq C\epsilon^{\frac{1}{4}}.$$

In order to estimate the remaining term in (11.20), we observe that for any $a \in L^p(0,t)$ and $1 \leq p, r \leq \infty$, such that $p^{-1} + r^{-1} = 1$, we have

$$\left| \int_0^t a(s) e^{-\frac{1}{\epsilon}\overline{i\lambda_{\epsilon,4}^{\tau,k}}(s-t)} ds \right| \leq C \int_0^t e^{-\frac{1}{\sqrt{\epsilon}}\text{Re}(i\lambda_1^{\tau,k})(s-t)} |a(s)| ds.$$

Direct calculations show that

$$\left\| e^{-\frac{1}{\sqrt{\epsilon}}\text{Re}(i\lambda_1^{\tau,k})(t-s)} \right\|_{L^r(0,t)} = \epsilon^{\frac{1}{2r}} \left[\frac{1}{-r\text{Re}(i\lambda_1^{\tau,k})} \left(e^{-\frac{r}{\sqrt{\epsilon}}\text{Re}(i\lambda_1^{\tau,k})t} - 1 \right) \right]^{\frac{1}{r}} e^{-\frac{1}{\sqrt{\epsilon}}\text{Re}(i\lambda_1^{\tau,k})t}.$$

Using the fact $\text{Re}(i\lambda_1^{\tau,k}) < 0$ again, we have

$$\left| \int_0^t a(s) e^{-\frac{1}{\epsilon}\overline{i\lambda_{\epsilon,4}^{\tau,k}}(s-t)} ds \right| \leq C \|a\|_{L^p(0,t)} \epsilon^{\frac{1}{2r}}. \quad (11.22)$$

Now applying $a(t)$ in (11.22) to $c_{1,\epsilon}^{\tau,k}$ and $c_{2,\epsilon}^{\tau,k}$, finally we get:

$$\tilde{b}_\epsilon^{\tau,k} \rightarrow 0, \quad \text{strongly in } L_{loc}^2(dt).$$

To finish the proof of the Proposition, we notice that

$$\int_{\Omega} \langle \tilde{g}_\epsilon, g_0^{\tau,k,\text{int}} \rangle dx = \tilde{b}_\epsilon^{\tau,k} + \int_{\Omega} \langle \tilde{g}_\epsilon, g_0^{\tau,k,\text{int}} - g_{\epsilon,4}^{\tau,k} \rangle dx.$$

Applying the error estimate (7.11) in Proposition 7.1, we finish the proof of the Proposition 11.2. \square

Consequently, we prove the Proposition 11.1.

11.5. Proof of Lemma 11.1.

Proof. Using the boundary condition of $g_{\varepsilon,4}^{\tau,k}$, namely (7.3), simple calculations yields that

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\partial\Omega} \langle \gamma \tilde{g}_\varepsilon \gamma g_{\varepsilon,4}^{\tau,k} (v \cdot n) \rangle d\sigma_x = \frac{1}{\varepsilon} \iint_{\Sigma_-} \gamma_- \tilde{g}_\varepsilon \gamma_- g_{\varepsilon,4}^{\tau,k} (v \cdot n) M dv d\sigma_x \\
& \quad + \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma_+ \tilde{g}_\varepsilon \left[(1 - \alpha_\varepsilon) L \gamma_- g_{\varepsilon,4}^{\tau,k} + \alpha_\varepsilon \langle \gamma_- g_{\varepsilon,4}^{\tau,k} \rangle_{\partial\Omega} + r_{\varepsilon,4}^k \right] (v \cdot n) M dv d\sigma_x \\
& = \frac{1}{\varepsilon} \iint_{\Sigma_-} \gamma_- g_{\varepsilon,4}^{\tau,k} [\gamma_- \tilde{g}_\varepsilon - (1 - \alpha_\varepsilon) L \gamma_+ \tilde{g}_\varepsilon - \alpha_\varepsilon \langle \gamma_+ \tilde{g}_\varepsilon \rangle_{\partial\Omega}] (v \cdot n) M dv d\sigma_x \\
& \quad + \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma_+ \tilde{g}_\varepsilon r_{\varepsilon,4}^k (v \cdot n) M dv d\sigma_x, \\
& = \frac{1}{\varepsilon} \iint_{\Sigma_+} (\gamma_+ \tilde{g}_\varepsilon - L \gamma_- \tilde{g}_\varepsilon) d\tilde{\nu}_\varepsilon - \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} (\gamma_+ \tilde{g}_\varepsilon - \langle \gamma_+ \tilde{g}_\varepsilon \rangle_{\partial\Omega}) d\tilde{\nu}_\varepsilon \\
& \quad + \frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma_+ \tilde{g}_\varepsilon r_{\varepsilon,4}^k (v \cdot n) M dv d\sigma_x,
\end{aligned} \tag{11.23}$$

where the measure $d\tilde{\nu}_\varepsilon = L \gamma_- g_{\varepsilon,4}^{\tau,k} (v \cdot n) M dv d\sigma_x$. From the boundary error estimate (7.12) in Proposition 7.1 (letting $r = \infty, p = 2$), and the fact that $\gamma_+ \tilde{g}_\varepsilon$ is bounded in $L^1(d\sigma_x, L^2(aM dv))$, it is easy to see that

$$\frac{1}{\varepsilon} \iint_{\Sigma_+} \gamma_+ \tilde{g}_\varepsilon r_{\varepsilon,4}^k (v \cdot n) M dv d\sigma_x \rightarrow 0, \quad \text{in } L^1_{loc}(dt),$$

as $\varepsilon \rightarrow 0$.

It remains to show that the first two terms in the righthand side of (11.23) go to zero as $\varepsilon \rightarrow 0$. It again follows from the *a priori* estimates Lemma 10.1 and 10.2. The main difficulty is $\alpha_\varepsilon/\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$, since $\alpha_\varepsilon = \sqrt{2\pi\chi}\sqrt{\varepsilon}$.

$$\begin{aligned}
& \frac{1}{\varepsilon} \iint_{\Sigma_+} (\gamma_+ \tilde{g}_\varepsilon - L \gamma_- \tilde{g}_\varepsilon) d\tilde{\nu}_\varepsilon - \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} (\gamma_+ \tilde{g}_\varepsilon - \langle \gamma_+ \tilde{g}_\varepsilon \rangle_{\partial\Omega}) d\tilde{\nu}_\varepsilon \\
& = \frac{1}{\varepsilon} \iint_{\Sigma_+} (\gamma_+ \tilde{g}_\varepsilon - L \gamma_- \tilde{g}_\varepsilon) d\tilde{\nu}_\varepsilon - \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} d\tilde{\nu}_\varepsilon \\
& \quad + \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \left(\frac{\langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2} - \frac{\langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \right) d\tilde{\nu}_\varepsilon \\
& \quad + \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \left(\left\langle \frac{\gamma_+ g_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \right\rangle_{\partial\Omega} - \frac{\langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}}{1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2} \right) d\tilde{\nu}_\varepsilon.
\end{aligned} \tag{11.24}$$

The renormalized boundary condition (10.23) yields that

$$\begin{aligned}
& \frac{1}{\varepsilon} (\gamma_+ \tilde{g}_\varepsilon - L \gamma_- \tilde{g}_\varepsilon) - \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \\
& = - \frac{\alpha_\varepsilon}{\varepsilon} \frac{\gamma_\varepsilon \varepsilon^2 \gamma_+ g_\varepsilon (\gamma_+ g_\varepsilon + \gamma_+ \hat{g}_\varepsilon)}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)} + \frac{\alpha_\varepsilon}{\varepsilon} \left(\frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2} - \frac{\gamma_\varepsilon}{1 + \varepsilon^2 \gamma_+ g_\varepsilon^2} \right).
\end{aligned} \tag{11.25}$$

Thus, after simple calculations, we have

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_{\partial\Omega} \langle \gamma \tilde{g}_\varepsilon \gamma g_{\varepsilon,4}^{\tau,k}(v \cdot n) \rangle d\sigma_x \\
&= -\frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \frac{\gamma_\varepsilon \varepsilon^2 \gamma_+ \hat{g}_\varepsilon (\gamma_+ g_\varepsilon + \gamma_+ \hat{g}_\varepsilon)}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)} L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n) M dv d\sigma_x \\
&+ \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \frac{\gamma_\varepsilon \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega} (\gamma_+ g_\varepsilon + \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega})}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2)} L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n) M dv d\sigma_x \\
&- \frac{\alpha_\varepsilon}{\varepsilon} \iint_{\Sigma_+} \left\langle \frac{\gamma_\varepsilon \varepsilon^2 \gamma_+ g_\varepsilon (\gamma_+ g_\varepsilon + \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega})}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \langle \gamma_+ g_\varepsilon \rangle_{\partial\Omega}^2)} \right\rangle_{\partial\Omega} L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n) M dv d\sigma_x.
\end{aligned} \tag{11.26}$$

The *a priori* estimates from boundary yields that all the three terms on the right-hand side of (11.26) are bounded in $L_{loc}^p(dt)$ for $p > 1$. In deed, the integral of the first term over $[t_1, t_2]$ is bounded by

$$\begin{aligned}
& \int_{t_1}^{t_2} \iint_{\Sigma_+} \sqrt{\frac{\alpha_\varepsilon}{\varepsilon}} \frac{\gamma_\varepsilon^{(1)}}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)^{1/4}} \frac{\sqrt{\alpha_\varepsilon}(\varepsilon \gamma_+ \hat{g}_\varepsilon)}{1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2} \frac{(\sqrt{\varepsilon} \gamma_+ g_\varepsilon + \sqrt{\varepsilon} \gamma_+ \hat{g}_\varepsilon)}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)^{3/4}} L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n) M dv d\sigma_x \\
&- \varepsilon \int_{t_1}^{t_2} \iint_{\Sigma_+} \frac{\alpha_\varepsilon}{\varepsilon^2} \gamma_\varepsilon^{(2)} \frac{\varepsilon \gamma_+ \hat{g}_\varepsilon (\varepsilon \gamma_+ g_\varepsilon + \varepsilon \gamma_+ \hat{g}_\varepsilon)}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)(1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2)} L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n) M dv d\sigma_x
\end{aligned} \tag{11.27}$$

Note that

$$\frac{\sqrt{\alpha_\varepsilon}(\varepsilon \gamma_+ \hat{g}_\varepsilon)}{1 + \varepsilon^2 \gamma_+ \hat{g}_\varepsilon^2} \quad \text{and} \quad \frac{(\sqrt{\varepsilon} \gamma_+ g_\varepsilon + \sqrt{\varepsilon} \gamma_+ \hat{g}_\varepsilon)}{(1 + \varepsilon^2 \gamma_+ g_\varepsilon^2)^{3/4}} \mathbf{1}_{\gamma_+ G_\varepsilon \leq 2 \langle G_\varepsilon \rangle_{\partial\Omega} \leq 4 \gamma_+ G_\varepsilon}$$

are bounded. Furthermore, $L_{\gamma_- g_{k,\varepsilon,2}^\pm}(v \cdot n)$ is bounded in $L^q((v \cdot n) M dv d\sigma_x)$ for $q \geq 2$. Now, the estimates (10.16), (10.18) of Lemma 10.2 imply that the first term in (11.27) is bounded in $L_{loc}^p(dt)$, the second term vanishes as $\varepsilon \rightarrow 0$. Similarly, using Lemma 10.1, Lemma 10.2, we can prove that the integrals over $[t_1, t_2]$ of the second and third terms of (11.26) can be decomposed into two terms, one is bounded in $L_{loc}^p(dt)$, the other vanishes in $L_{loc}^1(dt)$. Thus we proved the Lemma 11.1. \square

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